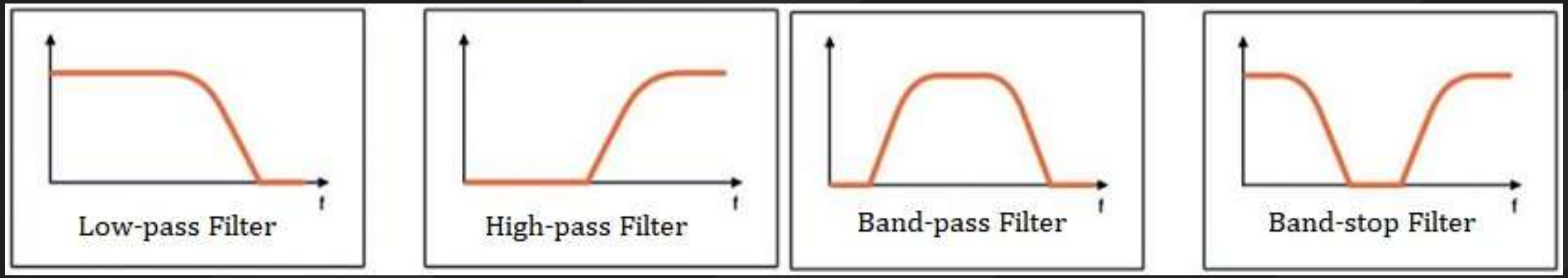
Finite Impulse Response (FIR) Filters

- What is a Filter?
  - Any medium through which the signal passes, whatever its form, can be regarded as a filter.
  - However, we do not usually think of something as a filter unless it can modify the signal in some way. For example, speaker wire is not considered a filter, but the speaker is
- A digital filter is just a filter that operates on digital signals, such as sound represented inside a computer.
- It is a computation which takes one sequence of numbers (the input signal) and produces a new sequence of numbers (the filtered output signal).

Types of Filter

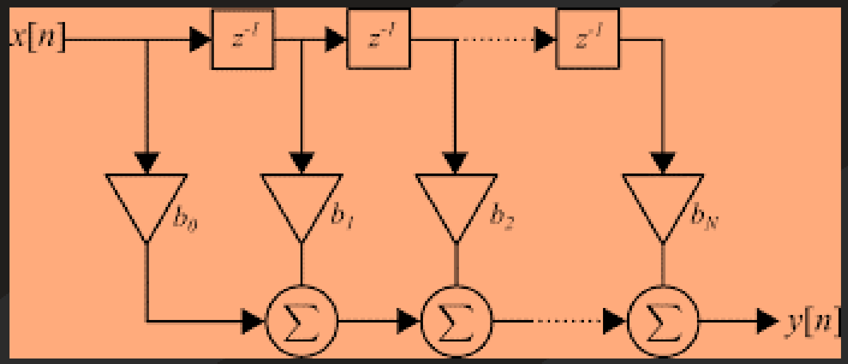


Finite Impulse Response (FIR) Filters

- It is one of two main types of digital filters used in DSP applications.
- FIR filter gets its name because the same number (finite) input values you get going into the filter, you get coming out of the output
- The design methods of FIR filter based on approximation of ideal filter
- Properties of FIR filter
  - *Require no feedback:* This means that any rounding errors are not compounded by summed iterations. The same relative error occurs in each calculation. This also makes implementation simpler.
  - *Inherent stability:* This is due to the fact that, because there is no required feedback, all the poles are located at the origin and thus are located within the unit circle (the required condition for stability in a Z transformed system).
  - *Phase Issue:* can easily be designed to be linear phase by making the coefficient sequence symmetric; linear phase, or phase change proportional to frequency, corresponds to equal delay at all frequencies. This property is sometimes desired for phase-sensitive applications, for example data communications, crossover filters, and mastering.
- The main disadvantage of FIR filters is that considerably more computation power

Finite Impulse Response (FIR) Filters

- A discrete-time FIR filter of order *N*. The top part is an N-stage delay line with *N + 1* taps. Each unit delay is a z^{-1} operator in the Z-transform notation.
- The output *y* of a linear time invariant system is determined by convolving its input signal *x* with its impulse response *b*.
- For a discrete-time FIR filter, the output is a weighted sum of the current and a finite number of previous values of the input.
- The operation is described by the following equation, which defines the output sequence *y[n]* in terms of its input sequence *x[n]*:



$$y(n) = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \dots + b_N x[n-N]$$

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

- $x(n)$: is the input sequence
- y(n) : is the output sequence
- b_k : filter coefficients that make up the impulse response
- N: is the filter order

FIR Impulse response

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k) \qquad h(n) = \sum_{k=0}^{N-1} b_k \delta[n-k]$$

The Z-transform of the impulse response yields the transfer function of the FIR filter

$$\mathrm{H}(z) = Z\{h(n)\}$$

$$= \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N-1} b_n z^{-n}$$

Linear phase FIR filter – Symmetric Impulse response

Let h(n) be an impulse response of a system then its Fourier transform can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad \text{------------------ (1)}$$

Since H($e^{j\omega n}$) is a complex value for linear phase FIR filter , then we can represent it in terms of magnitude and phase

$$H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{-j\alpha\omega} \quad \text{------------------ (2)}$$

Equating (1) & (2)

$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(e^{j\omega})|e^{-j\alpha\omega}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(e^{j\omega})|[\cos\alpha\omega - j\sin\alpha\omega]$$

Equating sin and cos terms

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n] = \pm|H(e^{j\omega})|[\cos\alpha\omega] \quad \text{----------------- (3)}$$

$$\sum_{n=0}^{N-1} h(n)[\sin\omega n] = \pm|H(e^{j\omega})|[\sin\alpha\omega] \quad \text{----------------- (4)}$$

(4) / (3)

$$\frac{\sum_{n=0}^{N-1} h(n)[\sin\omega n]}{\sum_{n=0}^{N-1} h(n)[\cos\omega n]} = \frac{\sin\alpha\omega}{\cos\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n)\sin\omega n\cos\alpha\omega = \sum_{n=0}^{N-1} h(n)\cos\omega n\sin\alpha\omega$$

$$0 = \sum_{n=0}^{N-1} h(n)[\cos\omega n\sin\alpha\omega - \sin\omega n\cos\alpha\omega]$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\sum_{n=0}^{N-1} h(n)[\sin(\alpha - n)\omega] = 0$$

The above equation will be zero when

$$h(n) = h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

Linear phase FIR filter – Symmetric Impulse response

The expression for Phase delay and group delay are

$$\tau_p = \frac{-\theta(\omega)}{\omega} \qquad \tau_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filter

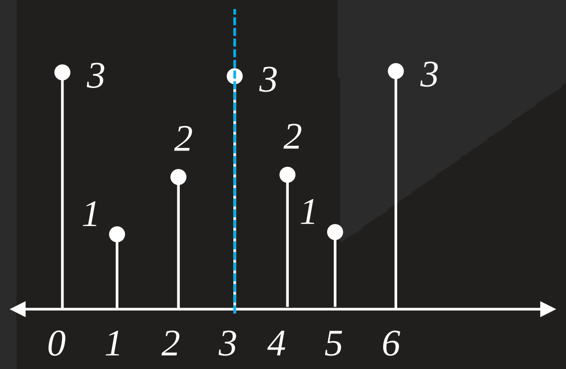
$$\theta(\omega) = -\alpha\omega\ , \qquad -\pi \le \omega \le \pi$$

$$h(n) = h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

From the equations and the conditions we can conclude that FIR filter will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

For N is odd



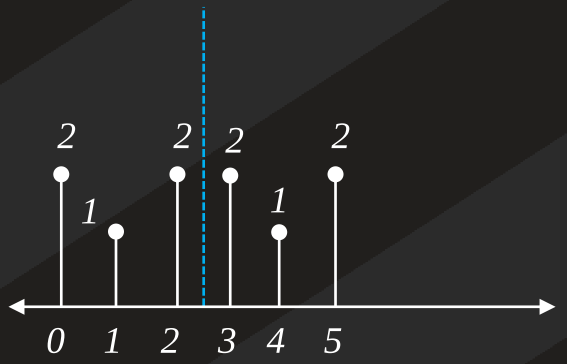
$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

$$h(n) = h(N-1-n)$$

$$h(5) = h(7-1-5)$$

$$= h(1)$$

For N is even



$$\alpha = \frac{N-1}{2} = \frac{6-1}{2} = 2.5$$

$$h(n) = h(N-1-n)$$

$$h(5) = h(6-1-5)$$

$$= h(0)$$

Linear phase FIR filter – Antisymmetric Impulse response

Let h(n) be an impulse response of a system then its Fourier transform can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad \text{------------------ (1)}$$

Since $H(e^{j\omega n})$ is a complex value for linear phase FIR filter , then we can represent it in terms of magnitude and phase. If only constant group delay is required

$$H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{-j(\beta-\alpha\omega)} \quad \text{------------------ (2)}$$

Equating (1) & (2)

$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(e^{j\omega})|e^{-j(\beta-\alpha\omega)}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(e^{j\omega})|[\cos(\beta-\alpha\omega) - j\sin(\beta-\alpha\omega)]$$

Equating sin and cos terms

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n] = \pm|H(e^{j\omega})|[\cos(\beta-\alpha\omega)] \quad \text{---------------- (3)}$$

$$\sum_{n=0}^{N-1} h(n)[\sin\omega n] = \pm|H(e^{j\omega})|[\sin(\beta-\alpha\omega)] \quad \text{---------------- (4)}$$

$$\theta(\omega) = \beta - \alpha\omega$$

(4) / (3)

$$\frac{\sum_{n=0}^{N-1} h(n)[\sin\omega n]}{\sum_{n=0}^{N-1} h(n)[\cos\omega n]} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n)\sin\omega n\cos(\beta-\alpha\omega) = \sum_{n=0}^{N-1} h(n)\cos\omega n\sin(\beta-\alpha\omega)$$

$$0 = \sum_{n=0}^{N-1} h(n)[\cos\omega n\sin(\beta-\alpha\omega) - \sin\omega n\cos(\beta-\alpha\omega)]$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\sum_{n=0}^{N-1} h(n)[\sin(\beta-(\alpha-n)\omega)] = 0$$

$$\beta = \frac{\pi}{2}$$

$$\sum_{n=0}^{N-1} h(n)[\cos(\alpha-n)\omega] = 0$$

The equation will be zero when

$$h(n) = -h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

Linear phase FIR filter – Asymmetric Impulse response

The expression for Phase delay and group delay are

$$\tau_p = \frac{-\theta(\omega)}{\omega} \qquad \tau_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filter

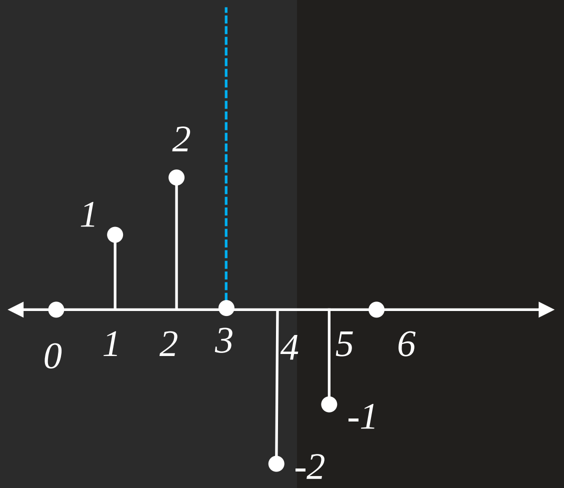
$$\theta(\omega) = \beta - \alpha\omega \;, \qquad -\pi \le \omega \le \pi$$

$$h(n) = -h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

From the equations and the conditions we can conclude that FIR filter will have constant group delay and not constant phase delay

For N is odd



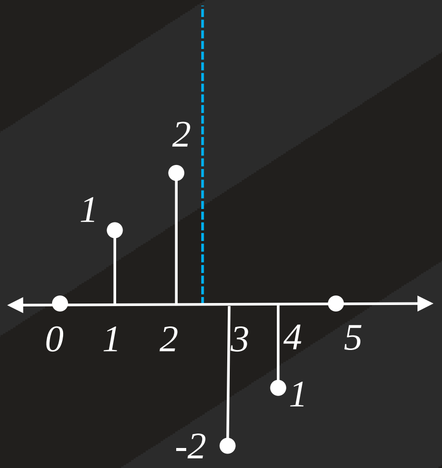
$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

$$h(n) = -h(N-1-n)$$

$$h(5) = -h(7-1-5)$$

$$= -h(1)$$

For N is even



$$\alpha = \frac{N-1}{2} = \frac{6-1}{2} = 2.5$$

$$h(n) = -h(N-1-n)$$

$$h(5) = -h(6-1-5)$$

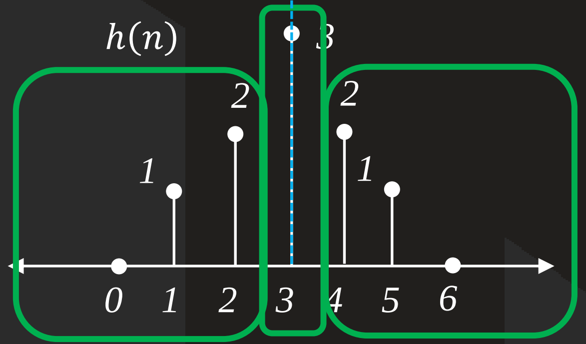
$$= h(0)$$

Frequency response of Linear phase FIR filters

Depending on the value of N and the type of symmetry of filter impulse response sequence there are mainly 4 types of linear phase FIR filter

- Symmetrical impulse response , N=odd
- Symmetrical impulse response , N=even
- Antisymmetric impulse response , N=odd
- Antisymmetric impulse response , N=even

Case1 : Symmetrical impulse response and N - odd



$N = 7$

$\alpha = \frac{N-1}{2} = 3$

Given h(n) and find the Fourier transform $H(e^{j\omega})$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

Since N is odd the centre of symmetry will be at n = $\frac{N-1}{2}$

Now lets split the equation into three parts

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

----------------- (1)

To arrange the limit we assume

$m = N - 1 - n$

$n = N - 1 - m$

When $n = \frac{N+1}{2}$

$\frac{N+1}{2} = N - 1 - m$

$m = \frac{N-3}{2}$

When $n = N - 1$

$N - 1 = N - 1 - m$

$m = 0$

Substitute in (1)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)e^{-j\omega(N-1-m)}$$

Put m=n

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

For symmetric impulse response $h(n) = h(N-1-n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega(N-1-n)}$$

Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega(N-1-n)}$$

Taking $e^{-j\omega\left(\frac{N-1}{2}\right)}$ outside

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{-j\omega n}.\,e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}.\,e^{j\omega\left(\frac{N-1}{2}\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$2\cos\theta = e^{j\theta} + e^{-j\theta}$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[2\cos\omega\left(\frac{N-1}{2}-n\right)\right]\right\}$$

Let

$$k = \frac{N-1}{2} - n$$

$$n = \frac{N-1}{2} - k$$

When $n = 0$

$$0 = \frac{N-1}{2} - k$$

$$k = \frac{N-1}{2}$$

When $n = \frac{N-3}{2}$

$$\frac{N-3}{2} = \frac{N-1}{2} - k$$

$$k = 1$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right)\cos\omega k\right\}$$

Put $k = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n\right\}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \qquad a(n) = 2h\left(\frac{N-1}{2}-n\right)$$

Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \quad a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

From this we can express the amplitude and phase function

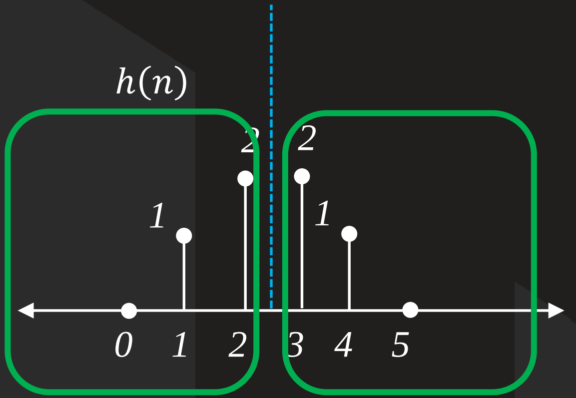
Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n$$

Phase

$$\angle H(e^{j\omega}) = -\omega\left(\frac{N-1}{2}\right) = -\alpha\omega$$

Case2 : Symmetrical impulse response and N - even



$N = 6$

$\alpha = \frac{N-1}{2} = 2.5$

Given h(n) and find the Fourier transform $H(e^{j\omega})$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

Since N is even the centre of symmetry will be at $n = \frac{N-1}{2}$

For symmetric impulse response with even number of samples and centre of symmetry lies between $n = \frac{N-2}{2}$ *and* $\frac{N}{2}$

Then we can split the equation into two parts

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \quad \text{----------------- (1)}$$

To arrange the limit we assume

$m = N - 1 - n$

$n = N - 1 - m$

When $n = \frac{N}{2}$

$\frac{N}{2} = N - 1 - m$

$m = \frac{N}{2} - 1$

When $n = N - 1$

$N - 1 = N - 1 - m$

$m = 0$

Substitute in (1)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{m=0}^{\frac{N-2}{2}} h(N-1-m)e^{-j\omega(N-1-m)}$$

Put m=n

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

For symmetric impulse response $h(n) = h(N-1-m)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} h(n)e^{-j\omega(N-1-n)}$$

Case2 : Symmetrical impulse response and N - even

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega(N-1-n)}$$

Taking $e^{-j\omega\left(\frac{N-1}{2}\right)}$ outside

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{-j\omega n}.e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}.e^{j\omega\left(\frac{N-1}{2}\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$$2\cos\theta = e^{j\theta} + e^{-j\theta}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[2\cos\omega\left(\frac{N-1}{2} - n\right)\right]\right\}$$

$$\frac{N-1}{2} - n = \frac{N}{2} - n - \frac{1}{2}$$

Let

$$k = \frac{N}{2} - n$$

$$n = \frac{N}{2} - k$$

When $n = 0$

$$0 = \frac{N}{2} - k$$

$$k = \frac{N}{2}$$

When $n = \frac{N-2}{2}$

$$\frac{N-2}{2} = \frac{N}{2} - k$$

$$k = 1$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - k\right)\cos\omega\left(k - \frac{1}{2}\right)\right\}$$

Put $k = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right)\cos\omega\left(n - \frac{1}{2}\right)\right\}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n - \frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2} - n\right)$$

Case2 : Symmetrical impulse response and N - even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2}-n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n-\frac{1}{2}\right)$$

Phase

$$\angle H(e^{j\omega}) = -\omega\left(\frac{N-1}{2}\right) = -\alpha\omega$$

Case 3 : Antisymmetrical impulse response and N - odd

In the similar way of symmetric, we get for antisymmetric as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{\frac{j\pi}{2}} \left\{ \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega(n) \right\}$$

Where

$$c(n) = 2h\left(\frac{N-1}{2} - n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$\left|H(e^{j\omega})\right| = \sum_{n=1}^{\frac{N-1}{2}} c(n) \cos \omega(n)$$

Phase

$$\angle H(e^{j\omega}) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega = \frac{\pi}{2} - \alpha\omega$$

Case 4 : Ant symmetrical impulse response and N - even

In the similar way of symmetric, we get for antisymmetric as

$$H\left(e^{j\omega}\right) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{\frac{j\pi}{2}} \left\{ \sum_{n=1}^{\frac{N}{2}} d(n) \sin\omega\left(n - \frac{1}{2}\right) \right\}$$

Where

$$d(n) = 2h\left(\frac{N}{2} - n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$\left|H\left(e^{j\omega}\right)\right| = \sum_{n=1}^{\frac{N}{2}} d(n) \sin\omega\left(n - \frac{1}{2}\right)$$

Phase

$$\angle H\left(e^{j\omega}\right) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega = \frac{\pi}{2} - \alpha\omega$$

SUMMARY

Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \quad a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

Case2 : Symmetrical impulse response and N – even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n - \frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2} - n\right)$$

Case 3 : Ant symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}e^{\frac{j\pi}{2}}\left\{\sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega(n)\right\}$$

Where

$$c(n) = 2h\left(\frac{N-1}{2} - n\right)$$

Case 4 : Ant symmetrical impulse response and N - even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}e^{\frac{j\pi}{2}}\left\{\sum_{n=1}^{\frac{N}{2}} d(n)\sin\omega\left(n - \frac{1}{2}\right)\right\}$$

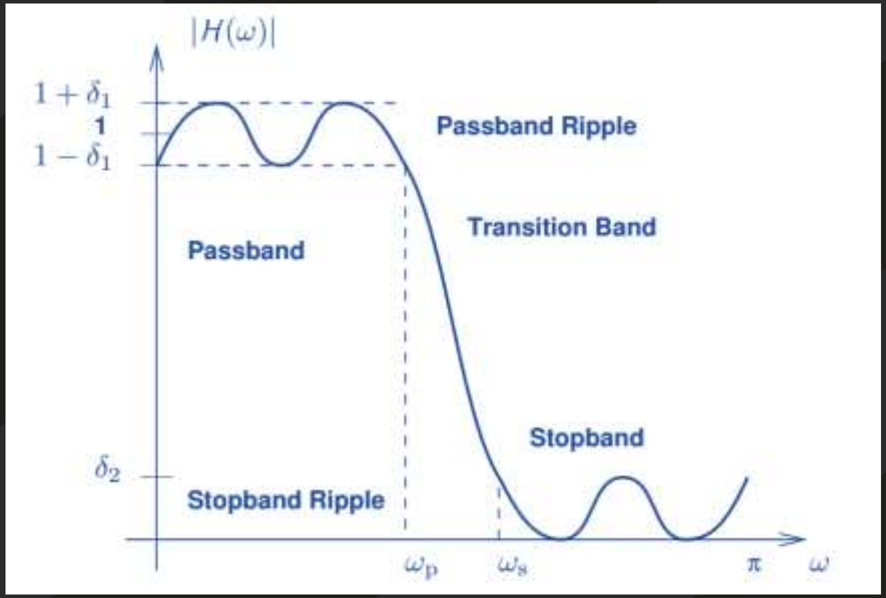
Where

$$d(n) = 2h\left(\frac{N}{2} - n\right)$$

Design of linear phase FIR filter

Why do we need a filter?

Frequency response of a practical lowpass filter



Digital filter design

1. ***Determining specification*** *: we need to know how strong the noise component is relative to the desired signal and how much we need to suppress the noise. This information is necessary to find the filter with minimum order for this application.*
2. ***Finding a transfer function*** *: we need to find a transfer function H(z) which will provide the required filtering.*
3. ***Choosing a realization structure*** *: there are many systems which can give the obtained transfer function and we must choose the appropriate one.*
4. ***Implementing the filter*** *: You have a couple of options for this step: a software implementation (such as a MATLAB or C code) or a hardware implementation (such as a DSP, a microcontroller, or an ASIC).*

Design of linear phase FIR filter : window method

Suppose that we want to design a lowpass filter with a cut off frequency of ω_c ,given frequency response

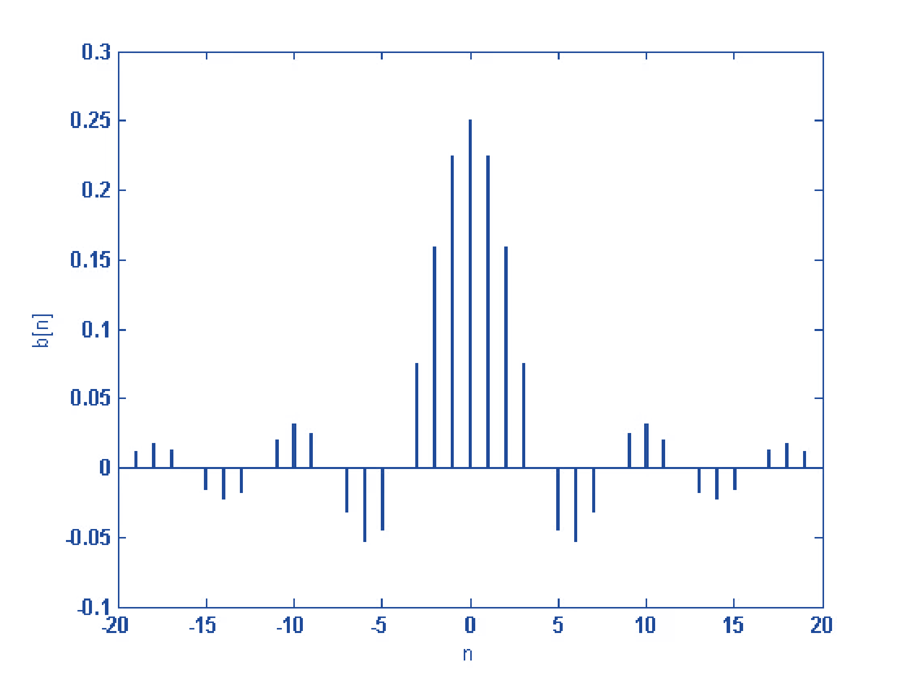
$$H_d(\omega) = \begin{cases} 1 \;, & |\omega| < \omega_c \\ 0 \;, & otherwise \end{cases}$$

To find the equivalent time-domain representation, we calculate the inverse discrete-time Fourier transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

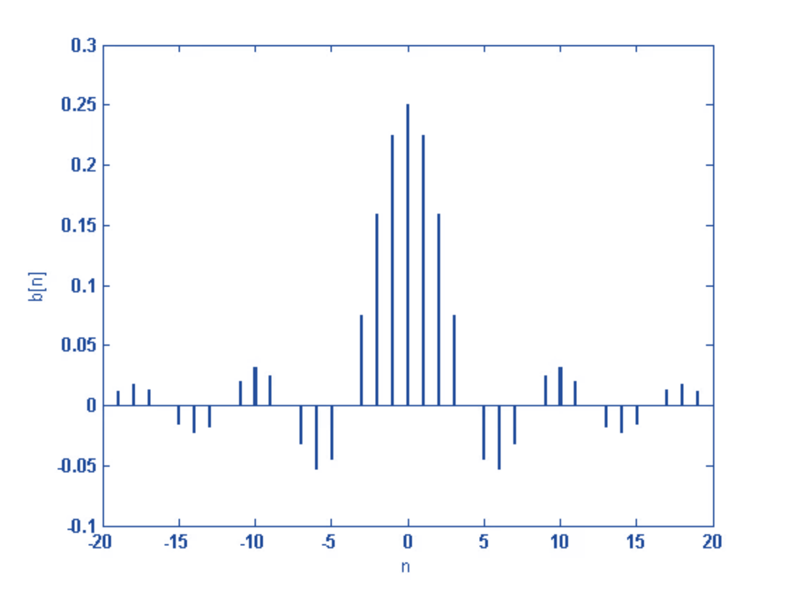
$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{\sin(n\omega_c)}{n\pi}$$

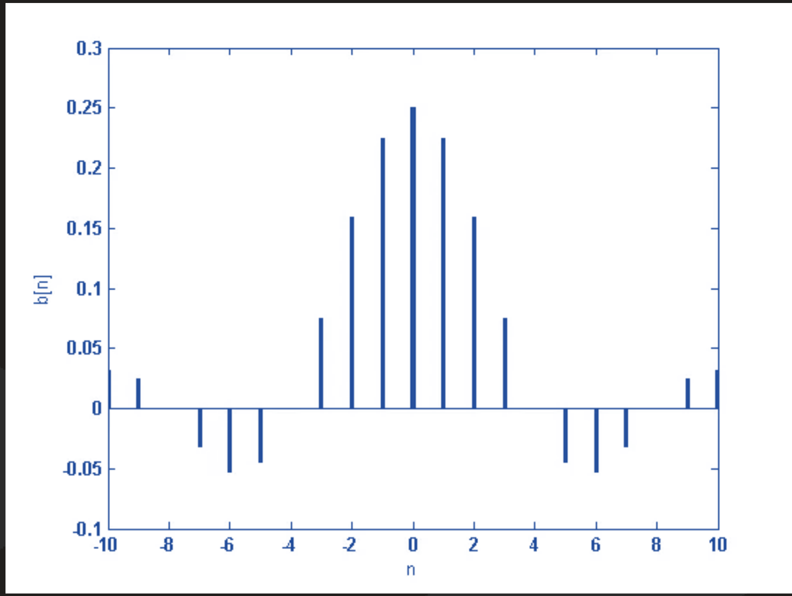


needs an infinite number of input samples to perform filtering and that the system is not a causal system. The solution will be to truncate the impulse response and use,

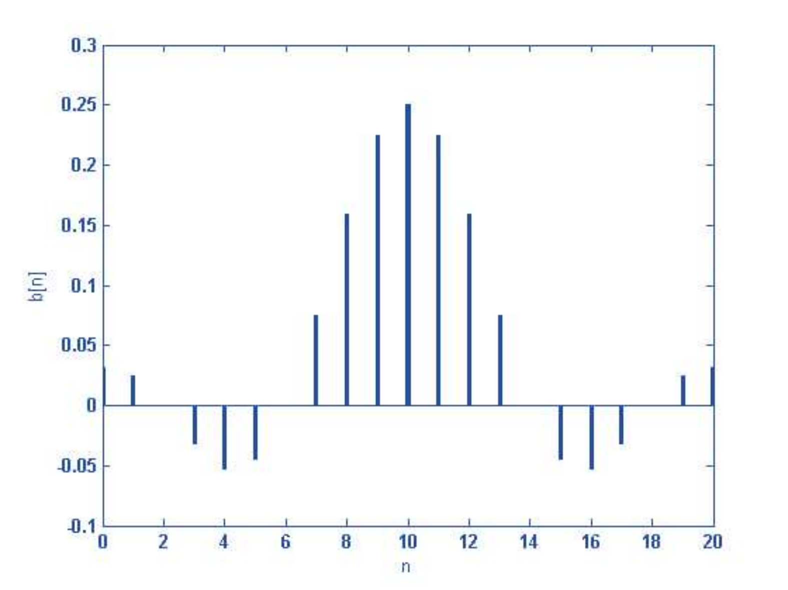
Design of linear phase FIR filter : window method



$h_d(n)$



Non causal system



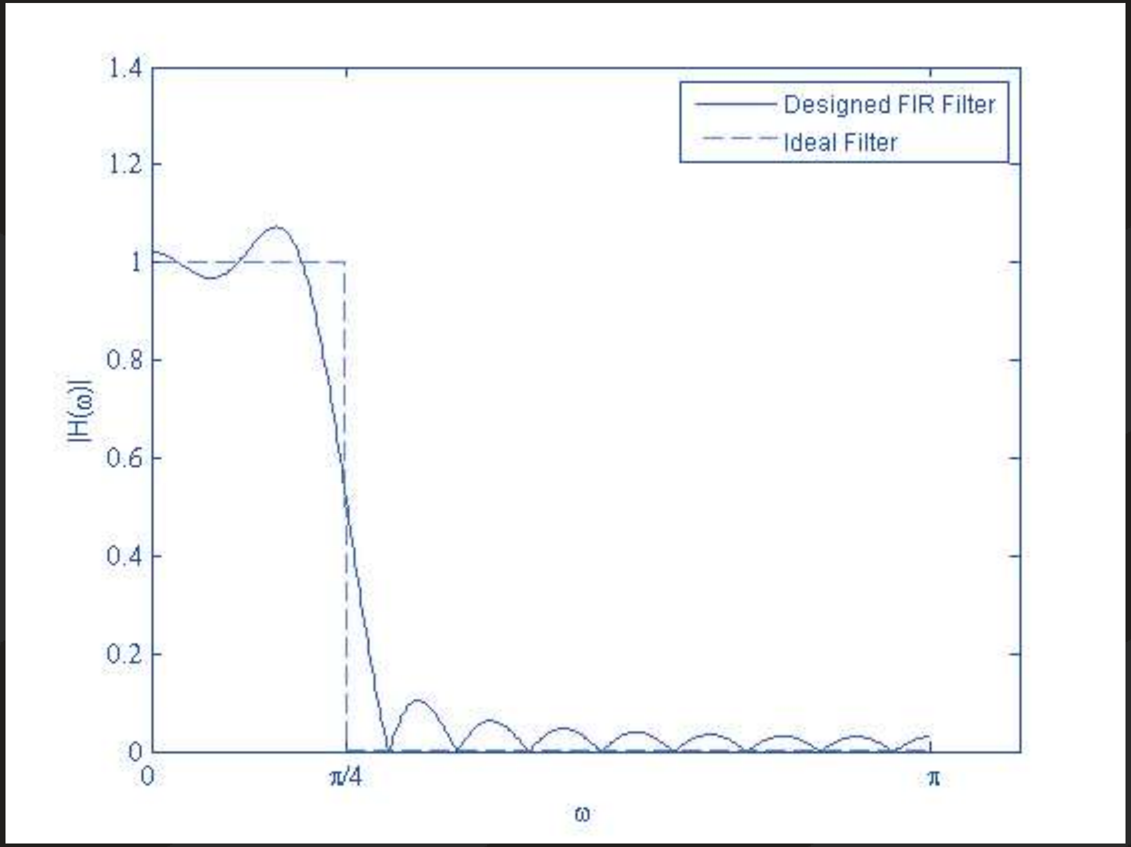
causal system & linear

but the system is delayed by $n = \frac{N-1}{2}$

There for considering an applied shift to $h_d(n)$ *and then multiplying with window function W(n)*

$$h(n) = h_d\left(n - \frac{N-1}{2}\right) * W(n)$$

Design of linear phase FIR filter : window method



Frequency response of the filter designed by a rectangular window

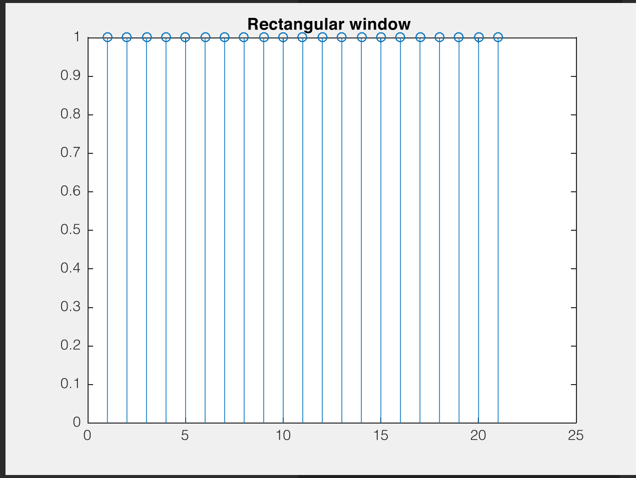
Design of linear phase FIR filter : window methods

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 1 & 0 \le n \le N \\ 0 & otherwise \end{cases}$$

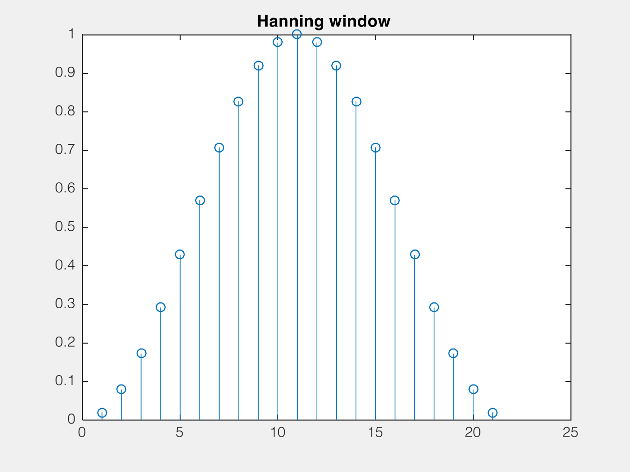


Hanning window

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right), & 0 \le n \le N \\ 0\ , & otherwise \end{cases}$$

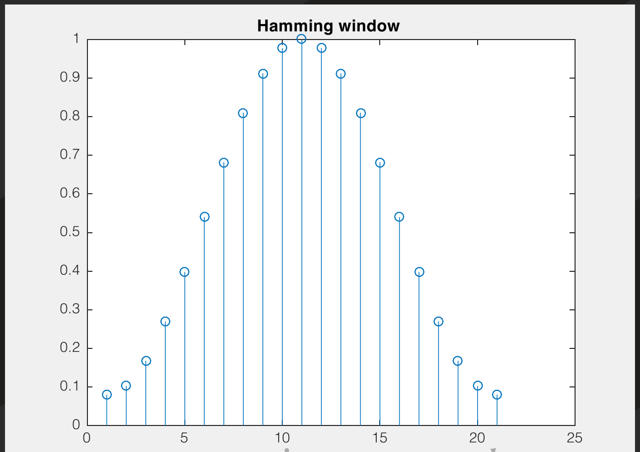


Hamming window

$$W_{Hm}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right), & 0 \le n \le N \\ 0\ , & otherwise \end{cases}$$



Design of linear phase FIR filter : window method

Design procedure

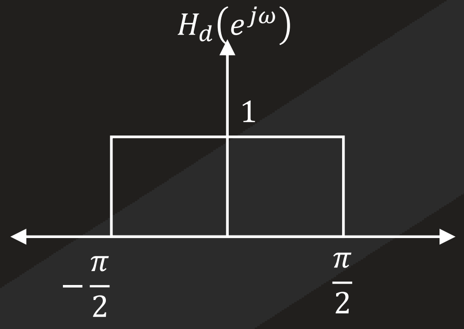
1. Choose desired frequency response of the filter $H_d(e^{j\omega})$
2. Take the invert Fourier transform of $H_d(e^{j\omega})$ to obtain $h_d(n)$
3. Choose a window sequence $W(n)$ and multiply it with $h_d(n)$ to convert infinite duration impulse response to finite duration impulse response

$$h(n) = h_d(n) * W(n)$$

4. The transfer function of the filter is obtained by taking Z-transform of $h(n)$

Q) Design an ideal lowpass filter with frequency response $H_d(e^{j\omega}) = \begin{cases} 1 & for\ -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 & otherwise \end{cases}$

find the value of h(n) for N=11 find H(z).



From the figure we know $\alpha = 0$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi jn}\left[e^{\frac{jn\pi}{2}} - e^{\frac{-jn\pi}{2}}\right]$$

$$= \frac{1}{2\pi jn}\left[2j\sin\frac{\pi n}{2}\right] \qquad = \frac{\sin\frac{\pi n}{2}}{n\pi}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Truncating $h_d(n)$ to 11 samples

$$h_d(n) = \begin{cases} \frac{\sin\frac{\pi n}{2}}{n\pi} & for = -5 \le n \le 5 \\ 0 & otherwise \end{cases}$$

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h(0) = \lim_{n\to 0}\frac{\sin\frac{\pi n}{2}}{n\pi} = \lim_{n\to 0}\frac{\sin\frac{\pi n}{2}}{\frac{n\pi}{2}2} = \frac{1}{2}$$

$$\lim_{n\to 0}\frac{\sin n}{n} = 1$$

for n=1

$$h(1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.318 = h(-1)$$

for n=2

$$h(2) = \frac{\sin\pi}{2\pi} = 0 = h(-2)$$

for n=3

$$h(3) = \frac{\sin\frac{3\pi}{2}}{3\pi} = \frac{-1}{3\pi} = -0.106 = h(-3)$$

for n=4

$$h(4) = \frac{\sin\frac{4\pi}{2}}{4\pi} = 0 = h(-4)$$

for n=5

$$h(5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = \frac{1}{5\pi} = 0.0636 = h(-5)$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} \quad = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^{5} + h(-4)Z^{4} + h(-3)Z^{3} + h(-2)Z^{2} + h(-1)Z^{1} + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$= h(0) + \sum_{n=1}^{5} h(n)[Z^{n} + Z^{-n}]$$

$$= 0.5 + 0.318(Z^{1} + Z^{-1}) + 0 - 0.106(Z^{3} + Z^{-3}) + 0 + 0.0636(Z^{5} + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)}H(Z)$$

$$= Z^{-5}[0.5 + 0.318(Z^{1} + Z^{-1}) - 0.106(Z^{3} + Z^{-3}) + 0.0636(Z^{5} + Z^{-5})]$$

$$H'(Z) = 0.0636 - 0.106Z^{-2} + 0.318Z^{-4} + 0.5Z^{-5} + 0.318Z^{-6} - 0.106Z^{-8} + 0.0636Z^{-10}$$

$h(0) = h(10) = 0.0636$

$h(1) = h(9) = 0$

$h(2) = h(8) = -0.106$

$h(3) = h(7) = 0$

$h(4) = h(6) = 0.318$

$h(5) = 0$

Design of linear phase FIR filter : window method

Design Steps

1. *Plot the desired frequency response* $H_d(e^{j\omega})$
2. *Determine the desired impulse response* $h_d(n)$ *by taking the inverse Fourier transform of* $H_d(e^{j\omega})$

$$\boldsymbol{h_d(n) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.\, e^{j\omega n} d\omega}$$

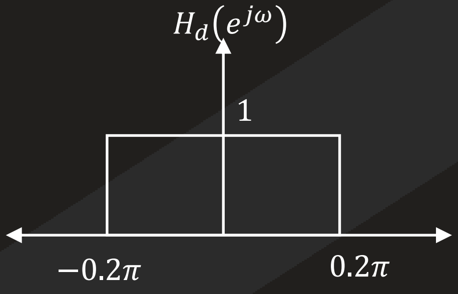
3. *Find the value of* $h_d(n)$ *for all 'n'*
4. *Choose a window sequence* $W(n)$ *and multiply it with* $h_d(n)$ *to get impulse response h(n)*

$$h(n) = h_d(n) * W(n)$$

5. *Take the Z – Transform of h(n)to get transfer function of the filter which is given and find coefficients*

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW



Q) Design a linear phase FIR low pass filter using rectangular window by taking 7 samples of window sequence and with a cut off frequency $\omega_c = 0.2\pi\ rad/sec$

or

Q) Design a linear phase FIR filter low pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for - \omega_c \le \omega \le \omega \\ 0 & otherwise \end{cases}$$ where $\omega_c = 0.2\pi$ and N=7

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} 1.e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi jn}\left[e^{j0.2\pi n} - e^{-j0.2\pi n}\right]$$

$$= \frac{1}{2\pi jn}[2j \sin 0.2\pi n] \quad = \frac{\sin 0.2\pi n}{n\pi}$$

Truncating $h_d(n)$ to 7 samples

$$h_d(n) = \begin{cases} \dfrac{\sin 0.2\pi n}{n\pi} & for = -3 \le n \le 3 \\ 0 & otherwise \end{cases}$$

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin 0.2\pi n}{n\pi} = \lim_{n\to 0}\frac{\sin 0.2\pi n}{\frac{n\pi}{0.2}\,0.2} = 0.2$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0}\frac{\sin n}{n} = 1$$

for n=1

$$h_d(1) = \frac{\sin 0.2\pi}{\pi} = 0.187 \quad = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 0.2\pi 2}{2\pi} = 0.151 \quad = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 0.2\pi 3}{3\pi} = 0.1009 \quad = h_d(-3)$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW

Now using rectangular window sequence W(n) and multiply $h_d(n)$ with it to get the impulse response *h(n)*

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

$$W_R(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & otherwise \end{cases}$$

Since $\alpha = 0$ *we get a non causal filter coefficient symmetrical about n=0*
so h(n) = h(-n)

for n=0 $\quad h(0) = h_d(0).W_R(0) \quad = 0.2$

for n=1 $\quad h(1) = h_d(1).W_R(1) \quad = 0.187 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_R(2) \quad = 0.1514 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_R(3) \quad = 0.1009 = h(-3)$

$$h(n) = [0.1009, 0.1514, 0.187, 0.2, 0.187, 0.1514, 0.1009]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} \quad = \sum_{n=-3}^{3} h(n)Z^{-n}$$

$$= h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3}$$

$$H(Z) = h(0) + \sum_{n=1}^{3} h(n)[Z^n + Z^{-n}]$$

$$= 0.2 + 0.187(Z^1 + Z^{-1}) + 0.151(Z^2 + Z^{-2}) + 0.1009(Z^3 + Z^{-3})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$= Z^{-3}[0.2 + 0.187(Z^1 + Z^{-1}) + 0.151(Z^2 + Z^{-2}) + 0.1009(Z^3 + Z^{-3})]$$

$$= 0.1009 + 0.151Z^{-1} + 0.187Z^{-2} + 0.2Z^{-3} + 0.187Z^{-4} + 0.151Z^{-5} + 0.1009Z^{-6}$$

$h(0) = h(6) = 0.1009 \qquad h(1) = h(5) = 0.151$

$h(2) = h(4) = 0.187 \qquad h(3) = 0.2$

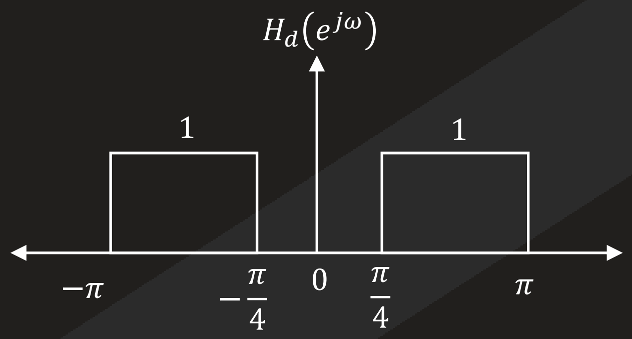
YouTube - IMPLearn

FIR FILTER DESIGN USING RECTANGULAR WINDOW

Q) Design a linear phase FIR high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Use rectangular window



From the figure we know $\alpha = 0$ also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ to 11 samples

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin\pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}4} \quad = 1 - \frac{1}{4} \quad = \frac{3}{4}$$

for n=1

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} \quad = -0.225 \quad = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} \quad = -0.1591 \quad = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} \quad = -0.075 \quad = h_d(-3)$$

for n=4

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} \quad = 0 \quad = h_d(-4)$$

for n=5

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} \quad = 0.045 \quad = h_d(-5)$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW

Now using rectangular window sequence $W_R(n)$ and multiply $h_d(n)$ with it to get the impulse response $h(n)$

$$W_R(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

Since $\alpha = 0$ we get a non causal filter coefficient symmetrical about n=0 so h(n) = h(-n)

for n=0 $\quad h(0) = h_d(0).W_R(0) \quad = 0.75$

for n=1 $\quad h(1) = h_d(1).W_R(1) \quad = -0.225 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_R(2) \quad = -0.1591 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_R(3) \quad = -0.075 = h(-3)$

for n=4 $\quad h(4) = h_d(4).W_R(4) \quad = 0 = h(-4)$

for n=5 $\quad h(5) = h_d(5).W_R(5) \quad = 0.0450 = h(-5)$

$$h(n) = [0.0450, 0, -0.075, -0.1591, -0.225, 0.75, -0.225 - 0.1591 - 0.0750, 0.0450]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} \quad = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{3} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.225(Z^1 + Z^{-1}) - 0.159(Z^2 + Z^{-2}) - 0.075(Z^3 + Z^{-3}) + 0.045(Z^5 + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)}H(Z)$$

$$= Z^{-5}[0.75 - 0.225(Z^1 + Z^{-1}) - 0.159(Z^2 + Z^{-2}) - 0.075(Z^3 + Z^{-3}) + 0.045(Z^5 + Z^{-5})]$$

$$= 0.045 - 0.075Z^{-2} - 0.159Z^{-3} - 0.225Z^{-4} + 0.75Z^{-5} - 0.225Z^{-6} - 0.1591Z^{-7} - 0.075Z^{-8} + 0.045Z^{-10}$$

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

$h(0) = h(10) = 0.045$ $\quad h(3) = h(7) = -0.159$

$h(1) = h(9) = 0$ $\quad h(4) = h(6) = -0.225$

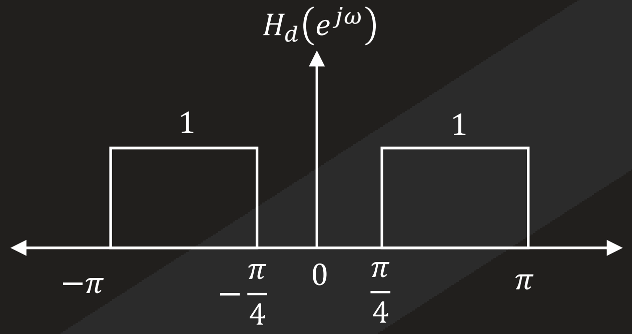
$h(2) = h(8) = -0.075$ $\quad h(5) = 0.75$

FIR FILTER DESIGN USING HANNING WINDOW

Q) Design a linear phase FIR filter high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Using Hanning window



From the figure we know $\alpha = 0$ also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

<u>*Solution*</u>

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ to 11 samples

Since for n=0 the equation becomes infinity so lets apply limit

<u>*for n=0*</u>

$$h_d(0) = \lim_{n\to 0}\frac{\sin\pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}4} = 1 - \frac{1}{4} = \frac{3}{4}$$

<u>*for n=1*</u>

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225 = h_d(-1)$$

<u>*for n=2*</u>

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} = -0.1591 = h_d(-2)$$

<u>*for n=3*</u>

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075 = h_d(-3)$$

<u>*for n=4*</u>

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} = 0 = h_d(-4)$$

<u>*for n=5*</u>

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045 = h_d(-5)$$

Hanning window

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 \;, & otherwise \end{cases}$$

Since $\alpha = 0$ *we get a non causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2n\pi}{10} & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

$$W_{Hn}(0) = 0.5 + 0.5 = 1$$

$$W_{Hn}(1) = 0.5 + 0.5\cos\frac{\pi}{5} = 09045 = W_{Hn}(-1)$$

$$W_{Hn}(2) = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655 = W_{Hn}(-2)$$

$$W_{Hn}(3) = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345 = W_{Hn}(-3)$$

$$W_{Hn}(4) = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945 = W_{Hn}(-4)$$

$$W_{Hn}(5) = 0.5 + 0.5\cos\frac{5\pi}{5} = 0 = W_{Hn}(-5)$$

FIR FILTER DESIGN USING HANNING WINDOW

Now using Hanning window sequence $W_{Hn}(n)$ and multiply $h_d(n)$ with it to get the impulse response *h(n)*

$$h(n) = h_d(n).W_{Hn}(n) \ for \ -5 \le n \le 5$$

for n=0 $h(0) = h_d(0).W_{Hn}(0) = 0.75(1) = 0.75$

for n=1 $h(1) = h_d(1).W_{Hn}(1) = -0.225(0.905) = -0.204 = h(-1)$

for n=2 $h(2) = h_d(2).W_{Hn}(2) = -0.159(0.655) = -0.104 = h(-2)$

for n=3 $h(3) = h_d(3).W_{Hn}(3) = -0.075(0.345) = -0.026 = h(-3)$

for n=4 $h(4) = h_d(4).W_{Hn}(4) = 0 = h(-4)$

for n=5 $h(5) = h_d(5).W_{Hn}(5) = 0 = h(-5)$

$$h(n) = [-0.026, -0.104, -0.204, 0.75, -0.204, -0.104, -0.026]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{5} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.204(Z^1 + Z^{-1}) - 0.104(Z^2 + Z^{-2}) - 0.026(Z^3 + Z^{-3})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$= Z^{-5}[0.75 - 0.204(Z^1 + Z^{-1}) - 0.104(Z^2 + Z^{-2}) - 0.026(Z^3 + Z^{-3})]$$

$$= -0.026Z^{-2} - 0.104Z^{-3} - 0.204Z^{-4} + 0.75Z^{-5} - 0.204Z^{-6} - 0.104Z^{-7} - 0.026Z^{-8}$$

$$h(0) = h(1) = h(9) = h(10) = 0$$

$h(2) = h(8) = -0.026$ $h(4) = h(6) = -0.204$

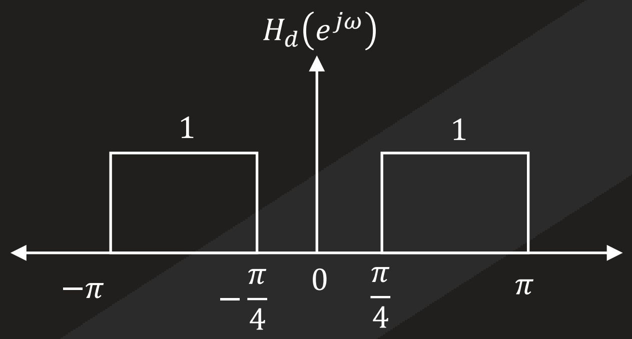
$h(3) = h(7) = -0.104$ $h(5) = 0.75$

FIR FILTER DESIGN USING HAMMING WINDOW

Q) Design a linear phase FIR filter high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Using Hamming window



From the figure we know $\alpha = 0$
also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ to 11 samples

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin \pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}\,4} = 1 - \frac{1}{4} = \frac{3}{4}$$

for n=1

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225 = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} = -0.1591 = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075 = h_d(-3)$$

for n=4

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} = 0 = h_d(-4)$$

for n=5

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045 = h_d(-5)$$

FIR FILTER DESIGN USING HAMMING WINDOW

Hamming window

$$W_H(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

Since $\alpha = 0$ *we get anon causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

$$W_H(n) = \begin{cases} 0.54 + 0.46\cos\frac{2n\pi}{10} & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

$$W_H(0) = 0.54 + 0.46 = 1$$

$$W_H(1) = 0.54 + 0.46\cos\frac{\pi}{5} = 0.912 = W_H(-1)$$

$$W_H(2) = 0.54 + 0.46\cos\frac{2\pi}{5} = 0.682 = W_H(-2)$$

$$W_H(3) = 0.54 + 0.46\cos\frac{3\pi}{5} = 0.398 = W_H(-3)$$

$$W_H(4) = 0.54 + 0.46\cos\frac{4\pi}{5} = 0.1678 = W_H(-4)$$

$$W_H(5) = 0.54 + 0.46\cos\frac{5\pi}{5} = 0.08 = W_H(-5)$$

FIR FILTER DESIGN USING HAMMING WINDOW

Now using Hamming window sequence $W_H(n)$ and multiply $h_d(n)$ with it to get the impulse response *h(n)*

$$h(n) = h_d(n).W_H(n) \quad for \ -5 \le n \le 5$$

for n=0 $\quad h(0) = h_d(0).W_H(0) = 0.75(1) = 0.75$

for n=1 $\quad h(1) = h_d(1).W_H(1) = -0.225(0.912) = -0.2056 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_H(2) = -0.159(0.682) = -0.1084 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_H(3) = -0.075(0.398) = -0.03 = h(-3)$

for n=4 $\quad h(4) = h_d(4).W_H(4) = 0 = h(-4)$

for n=5 $\quad h(5) = h_d(5).W_H(5) = -0.045(0.08) = 0.0036 = h(-5)$

$$h(n) = [0.0036, -0.03, -0.1084, -0.2056, 0.75, -0.2056, -0.1084, -0.03, 0.0036]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{5} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.2056(Z^1 + Z^{-1}) - 0.1084(Z^2 + Z^{-2}) - 0.03(Z^3 + Z^{-3}) + 0.0036(Z^5 + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)}H(Z)$$

$$= Z^{-5}[0.75 - 0.2056(Z^1 + Z^{-1}) - 0.1084(Z^2 + Z^{-2}) - 0.03(Z^3 + Z^{-3}) + 0.0036(Z^5 + Z^{-5})]$$

$$= 0.0036 - 0.03Z^{-2} - 0.1084Z^{-3} - 0.2052Z^{-4} + 0.75Z^{-5} - 0.2052Z^{-6} - 0.1084Z^{-7} - 0.03Z^{-8} + 00036Z^{-10}$$

$h(0) = h(10) = 0.0036$ $\quad h(3) = h(7) = -0.1084$

$h(1) = h(9) = 0$ $\quad h(4) = h(6) = -0.2052$

$h(2) = h(8) = -0.03$ $\quad h(5) = 0.75$

Design of linear phase FIR filter by frequency sampling technique

In this method the ideal frequency response is sampled at sufficient number of points these samples are the DFT coefficients of impulse response of filter. Hence impulse response of filter is determined by taking inverse DFT

Steps

1. Choose a desired frequency response $H_d\left(e^{j\omega}\right)$
2. Sample $H_d\left(e^{j\omega}\right)$ at N point by taking $\omega = \omega_k = \frac{2\pi k}{N}$ where k=0,1,2,3 … N-1
3. Compute the N samples of impulse response h(n) using the equation

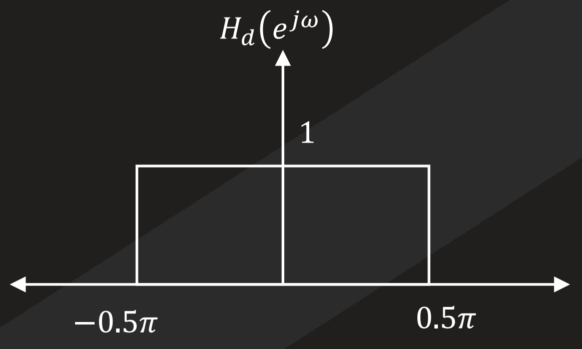
$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = even$$

4. Take Z-Transform of impulse response h(n) to get filter transfer function H(z)

$$H(Z) = \sum_{n=0}^{N-1} h(n)Z^{-n}$$

Design of linear phase FIR filter by frequency sampling technique

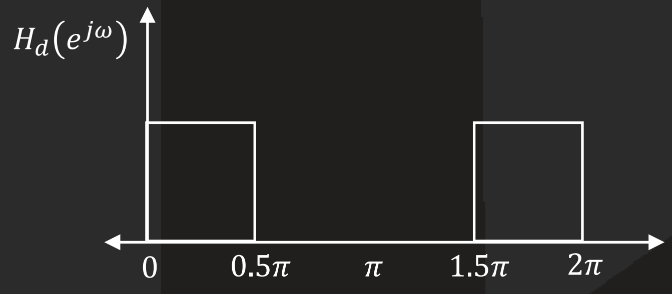


Q) Design a linear phase FIR low pass filter with cut off frequency of 0.5π rad/sample by taking 11 samples of ideal frequency response

Solution

Step1 : Find the desired frequency response

For digital sampling we are taking the limit as 0 to 2π



Due to symmetricity at (N-1)/2 then there will be an α exponential term in the expression

$$H_d(e^{j\omega}) = \begin{cases} 1.e^{-j\alpha\omega} & , 0 \le \omega \le 0.5\pi \\ 0 & , 0.5\pi \le \omega \le 1.5\pi \\ 1.e^{-j\alpha\omega} & , 1.5\pi \le \omega \le 2\pi \end{cases}$$

$$where, \alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

Step2: Sample $H_d(e^{j\omega})$ at N point by taking $\omega = \omega_k = \frac{2\pi k}{N}$

Sampling frequency $\omega_k = \frac{2\pi k}{11}$ *for k= 0 to 10*

for k=0 $\quad \omega_0 = \frac{2\pi * 0}{11} = 0$

for k=1 $\quad \omega_0 = \frac{2\pi * 1}{11} = 0.18\pi$

for k=2 $\quad \omega_0 = \frac{2\pi * 2}{11} = 0.36\pi$

for k=3 $\quad \omega_0 = \frac{2\pi * 3}{11} = 0.55\pi$

for k=4 $\quad \omega_0 = \frac{2\pi * 4}{11} = 0.73\pi$

for k=5 $\quad \omega_0 = \frac{2\pi * 5}{11} = 0.91\pi$

for k=6 $\quad \omega_0 = \frac{2\pi * 6}{11} = 1.09\pi$

for k=7 $\quad \omega_0 = \frac{2\pi * 7}{11} = 1.27\pi$

for k=8 $\quad \omega_0 = \frac{2\pi * 8}{11} = 1.45\pi$

for k=9 $\quad \omega_0 = \frac{2\pi * 9}{11} = 1.64\pi$

for k=10 $\quad \omega_0 = \frac{2\pi * 10}{11} = 1.82\pi$

$$H(k) = \begin{cases} e^{-j5\frac{2\pi k}{11}} & , for\ k = 0,1,2 \\ 0 & , for\ k = 3\ to\ 8 \\ e^{-j5\frac{2\pi k}{11}} & , for\ k = 9,10 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Step 3: Compute the N samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{5} Re\left[e^{-j5\frac{2\pi k}{11}}e^{\frac{j2\pi nk}{11}}\right]\right] \qquad = \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left[e^{-j5\frac{2\pi k}{11}}e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left[e^{j\frac{2\pi k}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2Re\left[e^{j\frac{2\pi}{11}(n-5)}\right] + 2Re\left[e^{j\frac{4\pi}{11}(n-5)}\right]\right]$$

$$h(n) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(n-5)\right) + 2\cos\left(\frac{4\pi}{11}(n-5)\right)\right]$$

$$H(k) = \begin{cases} e^{-j5\frac{2\pi k}{11}} & , for\ k = 0,1,2 \\ 0 & , for\ k = 3\ to\ 8 \\ e^{-j5\frac{2\pi k}{11}} & , for\ k = 9,10 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Now let's calculate h(n) for n = 0 to 10, using symmetric condition (h(n)=h(N-1-m))

for n=0

$$h(0) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(0-5)\right) + 2\cos\left(\frac{4\pi}{11}(0-5)\right)\right] = 0.0694$$

for n=1

$$h(1) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(1-5)\right) + 2\cos\left(\frac{4\pi}{11}(1-5)\right)\right] = -0.054$$

for n=2

$$h(2) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(2-5)\right) + 2\cos\left(\frac{4\pi}{11}(2-5)\right)\right] = -0.1094$$

for n=3

$$h(3) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(3-5)\right) + 2\cos\left(\frac{4\pi}{11}(3-5)\right)\right] = 0.0473$$

for n=4

$$h(4) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(4-5)\right) + 2\cos\left(\frac{4\pi}{11}(4-5)\right)\right] = 0.3194$$

for n=5

$$h(5) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(5-5)\right) + 2\cos\left(\frac{4\pi}{11}(-5)\right)\right] = 0.4595$$

for n=6

$$h(6) = h(11-1-6) = h(4) = 0.3194$$

for n=7

$$h(7) = h(11-1-7) = h(3) = 0.0473$$

for n=8

$$h(8) = h(11-1-8) = h(2) = -0.1094$$

for n=9

$$h(9) = h(11-1-9) = h(1) = -0.054$$

for n=10

$$h(10) = h(11-1-10) = h(0) = 0.0694$$

Design of linear phase FIR filter by frequency sampling technique

Step 4: Take Z-Transform of impulse response h(n) to get filter transfer function H(z)

$$H(Z) = \sum_{n=0}^{10} h(n)Z^{-n}$$

$$= h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5} + h(6)Z^{-6} + h(7)Z^{-7} + h(8)Z^{-8} + h(9)Z^{-9} + h(10)Z^{-10}$$

$$H(Z) = 0.0694(1 + Z^{-10}) - 0.054(Z^{-1} + Z^{-9}) - 0.1094(Z^{-2} + Z^{-8}) + 0.0473(Z^{-3} + Z^{-7}) + 0.3194(Z^{-4} + Z^{-6}) + 0.4595Z^{-5}$$

$h(6) = h(11 - 1 - 6) = h(4) \quad = 0.3194$

$h(7) = h(11 - 1 - 7) = h(3) \quad = 0.0473$

$h(8) = h(11 - 1 - 8) = h(2) \quad = -0.1094$

$h(9) = h(11 - 1 - 9) = h(1) \quad = -0.054$

$h(10) = h(11 - 1 - 10) = h(0) \quad = 0.0694$

Design of linear phase FIR filter by frequency sampling technique

Q) Using frequency sampling method, design a band pass filter with the following specifications , sampling frequency *F=8000Hz* ,cut off frequency f_{c1}*=1000Hz*, f_{c2}*=3000Hz*, Determine the filter coefficients for N=7

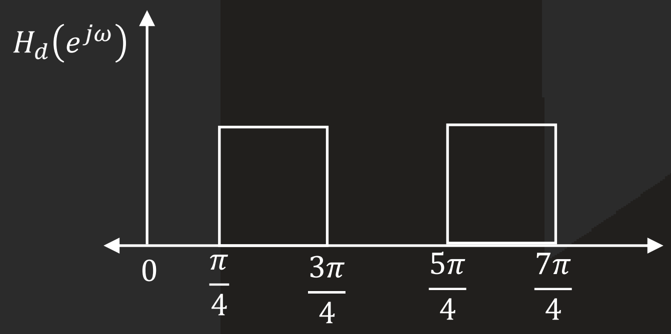
Solution

Step1 : Find the desired frequency response

For digital sampling we are taking the limit as 0 to 2π

$$\omega_{c1} = 2\pi f_{c1}T = \frac{2\pi f_{c1}}{F} = \frac{2\pi 1000}{8000} = \frac{\pi}{4} = 0.25\pi$$

$$\omega_{c2} = 2\pi f_{c2}T = \frac{2\pi f_{c2}}{F} = \frac{2\pi 3000}{8000} = \frac{3\pi}{4} = 0.75\pi$$



Due to symmetricity at (N-1)/2 then there will be an α exponential term in the expression

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & ,0.25\pi \le \omega \le 0.75\pi \\ 0 & ,0.75\pi \le \omega \le 1.25\pi \\ e^{-j\alpha\omega} & ,1.25\pi \le \omega \le 1.7\pi \end{cases}$$

$$where, \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

Step2: Sample $H_d(e^{j\omega})$ *at N point by taking*

$$\omega = \omega_k = \frac{2\pi k}{N}$$

Sampling frequency $\omega_k = \frac{2\pi k}{7}$ *for k= 0 to 6*

for k=0 $\omega_0 = \frac{2\pi * 0}{7} = 0$

for k=1 $\omega_0 = \frac{2\pi * 1}{7} = 0.28\pi$

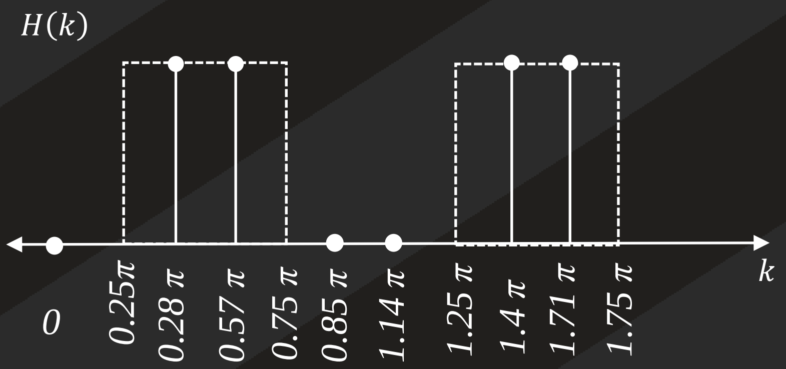
for k=2 $\omega_0 = \frac{2\pi * 2}{7} = 0.57\pi$

for k=3 $\omega_0 = \frac{2\pi * 3}{7} = 0.85\pi$

for k=4 $\omega_0 = \frac{2\pi * 4}{7} = 1.14\pi$

for k=5 $\omega_0 = \frac{2\pi * 5}{7} = 1.4\pi$

for k=6 $\omega_0 = \frac{2\pi * 6}{7} = 1.71\pi$



$$H(k) = \begin{cases} 0 & ,for\ k = 0 \\ e^{-j3\frac{2\pi k}{7}} & ,for\ k = 1,2 \\ 0 & ,for\ k = 3,4 \\ e^{-j3\frac{2\pi k}{7}} & ,for\ k = 5,6 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Step 3: Compute the N samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{7}\left[0 + 2\sum_{k=1}^{3} Re\left[e^{-j3\frac{2\pi k}{7}} e^{\frac{j2\pi nk}{7}}\right]\right] \qquad = \frac{1}{7}\left[2\sum_{k=1}^{2} Re\left[e^{-j3\frac{2\pi k}{7}} e^{\frac{j2\pi nk}{7}}\right]\right]$$

$$= \frac{2}{7}\left[\sum_{k=1}^{2} Re\left[e^{j\frac{2\pi k}{7}(n-3)}\right]\right]$$

$$= \frac{2}{7}\left[2Re\left[e^{-j\frac{2\pi}{7}(n-3)}\right] + 2Re\left[e^{-j\frac{4\pi}{7}(n-3)}\right]\right]$$

$$h(n) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(n-3)\right) + 2\cos\left(\frac{4\pi}{7}(n-3)\right)\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$H(k) = \begin{cases} 0 & , for\ k = 0 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 1,2 \\ 0 & , for\ k = 3,4 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 5,6 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Now let's calculate h(n) for n = 0 to 6, using symmetric condition (h(n)=h(N-1-n))

for n=0

$$h(0) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(0-3)\right) + 2\cos\left(\frac{4\pi}{7}(0-3)\right)\right] = -0.0792$$

for n=1

$$h(1) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(1-3)\right) + 2\cos\left(\frac{4\pi}{7}(1-3)\right)\right] = -0.321$$

for n=2

$$h(2) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(2-3)\right) + 2\cos\left(\frac{4\pi}{7}(2-3)\right)\right] = 0.1145$$

for n=3

$$h(3) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(3-3)\right) + 2\cos\left(\frac{4\pi}{7}(3-3)\right)\right] = 0.571$$

for n=4

$$h(4) = h(7-1-4) = h(2) = 0.1145$$

for n=5

$$h(5) = h(7-1-5) = h(1) = -0.321$$

for n=6

$$h(6) = h(7-1-6) = h(0) = -0.0792$$

Design of linear phase FIR filter by frequency sampling technique

Step 4: Take Z-Transform of impulse response h(n) to get filter transfer function H(z)

$$H(Z) = \sum_{n=0}^{6} h(n) Z^{-n}$$

$$= h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5} + h(6)Z^{-6}$$

$$H(Z) = -0.0792(1 + Z^{-6}) - 0.321(Z^{-1} + Z^{-5}) + 0.1145(Z^{-2} + Z^{-4}) + 0.571Z^{-3}$$

$$h(6) = h(7-1-6) = h(0) = -0.0792$$

$$h(5) = h(7-1-5) = h(1) = -0.321$$

$$h(4) = h(7-1-4) = h(2) = 0.1145$$

$$h(3) = 0.571$$

Infinite Impulse Response (IIR) Filters

- *The FIR filters are non recursive type filters(present input depends on the present and previous inputs) where as IIR filters are recursive type (present input depends on the present, past and output samples)*
- *IIR (infinite impulse response) filters are generally chosen for applications where linear phase is not too important and memory is limited.*
- *They have been widely deployed in audio equalization, biomedical sensor signal processing, IoT/IIoT smart sensors and high-speed telecommunication/RF applications*
- *IIR filter have infinite-duration impulse responses, hence they can be matched to analog filters, all of which generally have infinitely long impulse responses.*
- *The basic techniques of IIR filter design transform well-known analog filters into digital filters using complex-valued mappings.*
- *First we design an antilog prototype filter and then transform the prototype to a digital filter, hence it is also called indirect method*

Infinite Impulse Response (IIR) Filters

- *An IIR filter is categorized by its theoretically infinite impulse response, Practically speaking, it is not possible to compute the output of an IIR using this equation. Therefore, the equation may be re-written in terms of a finite number of poles p and zeros q, as defined by the linear constant coefficient difference equation*

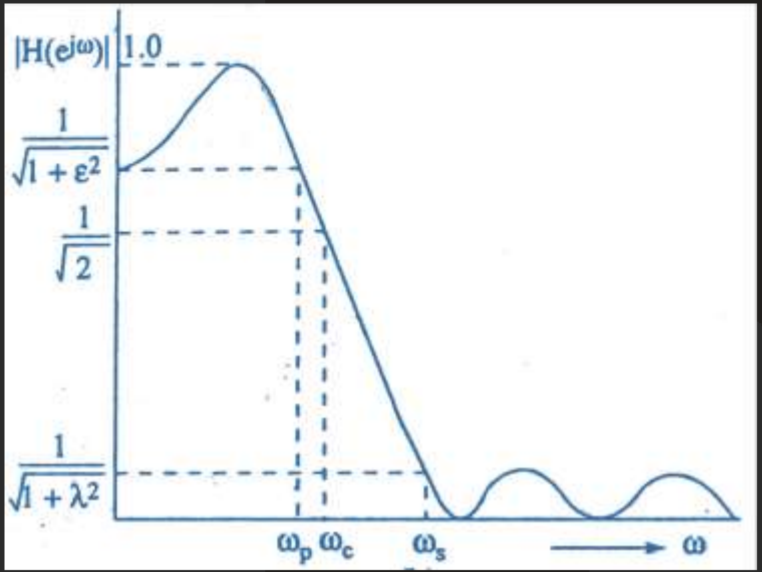
$$y(n) = \sum_{k=0}^{q} b_k x(n-k) - \sum_{l=1}^{p} a_k y(n-k)$$

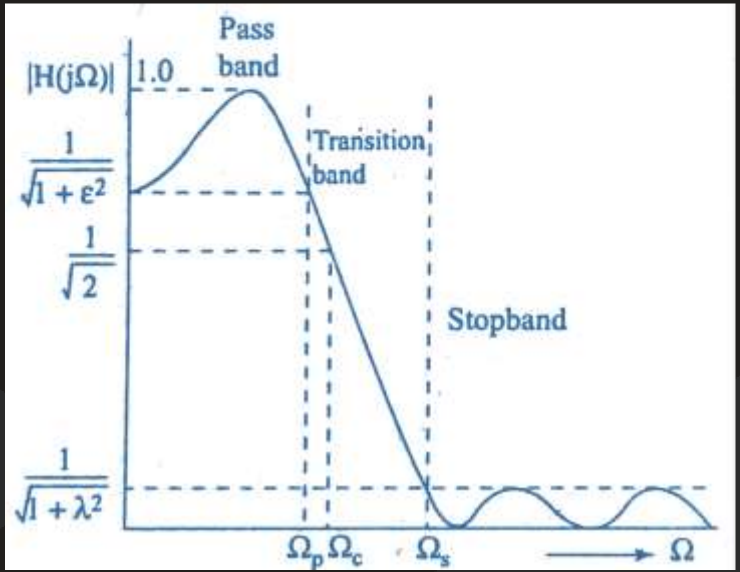
- *where, a(k) and b(k) are the filter's denominator and numerator polynomial coefficients, who's roots are equal to the filter's poles and zeros respectively. Thus, a relationship between the difference equation and the z-transform (transfer function) may therefore be defined by using the z-transform delay property such that,*

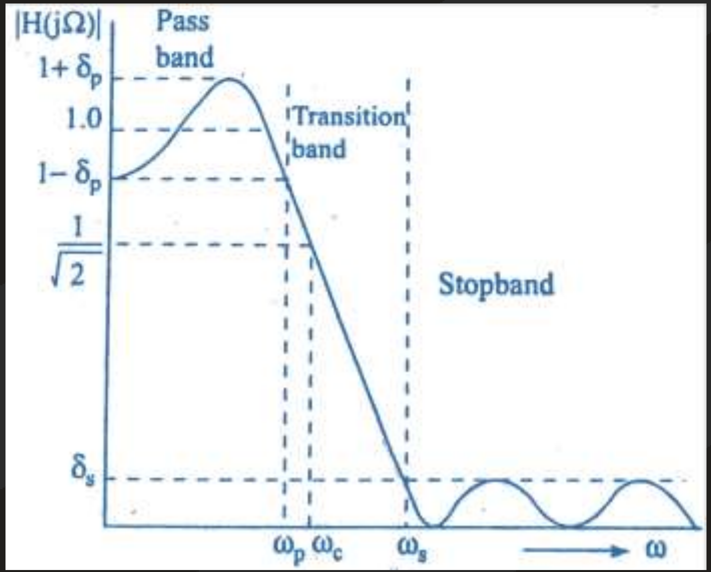
$$H(z) = \sum_{n=0}^{\infty} y(n) Z^{-n} = \frac{\sum_{k=0}^{q} b_k Z^{-k}}{1 + \sum_{k=1}^{p} a_k Z^{-k}}$$

- *As seen, the transfer function is a frequency domain representation of the filter.*
- *Notice also that the poles act on the output data, and the zeros on the input data.*
- *Since the poles act on the output data, and affect stability, it is essential that their radii remain inside the unit circle (i.e. <1) for BIBO (bounded input, bounded output) stability. The radii of the zeros are less critical, as they do not affect filter stability.*

Specifications for magnitude response of lowpass filter







Digital

ω_p → *Passband frequency (rad/samples)*

ω_s → *Stopband frequency (rad/samples)*

ω_c → *3dB cut off frequency (rad/samples)*

ε → *Passband parameter*

λ → *Stopband parameter*

Analog

Ω_p → *Passband frequency (rad/sec)*

Ω_s → *Stopband frequency (rad/sec)*

Ω_c → *3dB cut off frequency (rad/sec)*

δ_p → *Passband error tolerance*

δ_s → *max allowable magnitude in stop band*

Alternate specifications of lowpass filter

$$\varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p}$$

$$\lambda = \frac{\sqrt{\left(1+\delta_p\right)^2 - \delta_s^2}}{\delta_s}$$

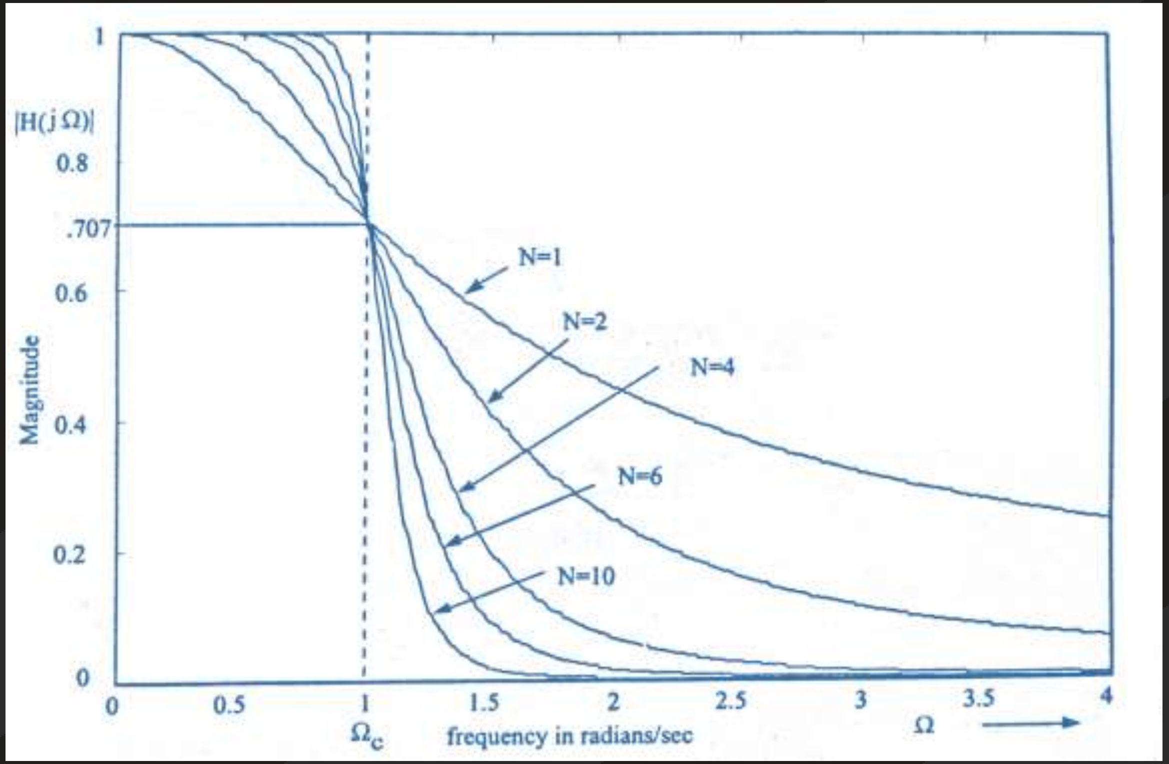
Design of steps of IIR Filters

1. *Map the desired digital filter specifications into those for an equivalent analog filter*
2. *Derive the analog transfer function for the analog prototype*
3. *Transform the transfer function of the analog prototype into an equivalent digital filter transfer function*

Analog lowpass filter design

Mainly there are two types of analog filter designs

1. *Butterworth Filter*
2. *Chebyshev filter*



Analog low pass Butterworth Filter

The magnitude function of the lowpass Butterworth filter is

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}}$$

Where N is the order of the filter

Properties of Butterworth filters

1. *Butterworth filters are all pole design*
2. *The magnitude of normalised Butterworth filter is 1/√2 at cut off frequency Ω_c*
3. *The filter order specified the filter*
4. *Magnitude is maximally flat at the origin*
5. *As N increases the response approaches to ideal response*

Analog low pass Butterworth Filter

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)\left(s^2+\sqrt{2s}+1\right)(s^2+0.517s+1)}$

Order of the Butterworth filter

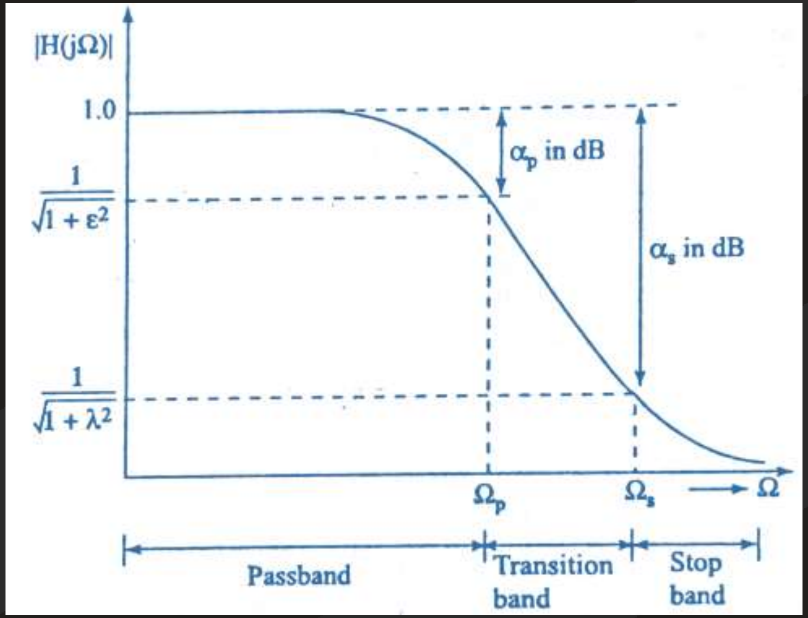
Let the maximum passband attenuation in positive dB is α_p (<3dB) at passband frequency Ω_p and α_s is the minimum stopband attenuation at stopband frequency Ω_s . The magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]^{\frac{1}{2}}}$$

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

Taking log on both sides

$$20 \log H(j\Omega) = 10 \log 1 - 10 \log(1 + \varepsilon^2)$$



Ω_p → *Passband frequency (rad/sec)*

Ω_s → *Stopband frequency (rad/sec)*

ε → *Passband parameter*

λ → *Stopband parameter*

α_p → *Passband attenuation*

α_s → *Stopband attenuation*

Order of the Butterworth filter

At $\Omega = \Omega_p$, $20 \log H(j\Omega) = -\alpha_p$

$$\alpha_p = 10 \log(1 + \varepsilon^2)$$

$$0.1\alpha_p = \log(1 + \varepsilon^2)$$

Taking antilog on both sides

$$10^{0.1\alpha_p} = 1 + \varepsilon^2$$

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2}}$$

At $\Omega = \Omega_s$, $20 \log H(j\Omega) = -\alpha_s$

$$\alpha_s = 10 \log\left(1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

Taking antilog on both sides

$$10^{0.1\alpha_s} - 1 = \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{\varepsilon^2}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

Taking log on both sides and finding the value of N

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

or

$$N \geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

Steps to design an analog Butterworth filter

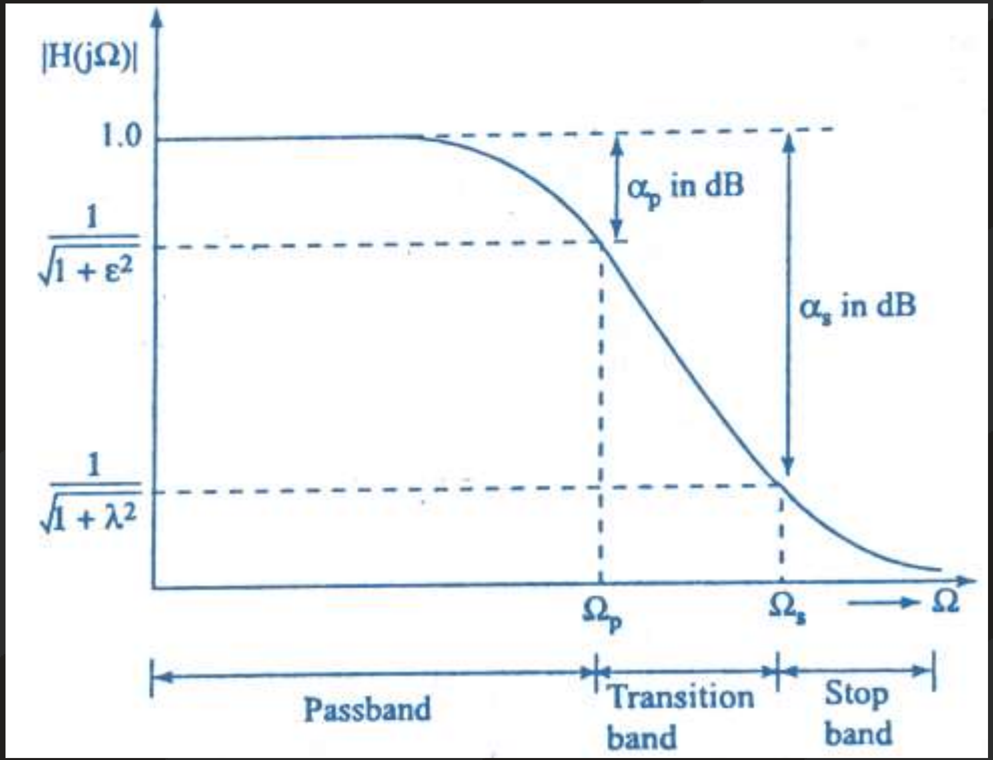
1. *Find the order of the filter N &round off to higher integer*
2. *Find the transfer function H(s) for Ω_c=1 rad/sec for the values of N*
3. *Calculate value of cut-off frequency Ω_c*
4. *Find transfer function Ha(s) for the value of Ω_c calculated by substituting $s = \frac{s}{\Omega_c}$ in H(s)*

Design an analog Butterworth filter

Q) For given specifications design an analog Butterworth filter

$$0.9 \leq |H(j\Omega)| \leq 1 \; for \; 0 \leq \Omega \leq 0.2\pi$$

$$|H(j\Omega)| \leq 0.2 \; for \; 0.4\pi \leq \Omega \leq \pi$$



<u>*Solution*</u>

$$\Omega_p = 0.2\pi \qquad \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \qquad \varepsilon = 0.484$$

$$\Omega_s = 0.4\pi \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \qquad \lambda = 4.898$$

Step 1: Find the order of the filter N & round off to higher integer

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34 \qquad N \approx 4$$

Design an analog Butterworth filter

Step 2: Find the transfer function H(s) for Ωc=1 rad/sec for the values of N

$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2}s+1)(s^2+0.517s+1)}$

Step 3: Calculate value of cut-off frequency Ωc

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2N}}} = \frac{0.2\pi}{\varepsilon^{\frac{1}{4}}} = 0.24\pi$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2}}$$

Design an analog Butterworth filter

Step 4: Find transfer function H(s) for the value of Ω_c calculated by substituting $s = \frac{s}{\Omega_c}$ in H(s)

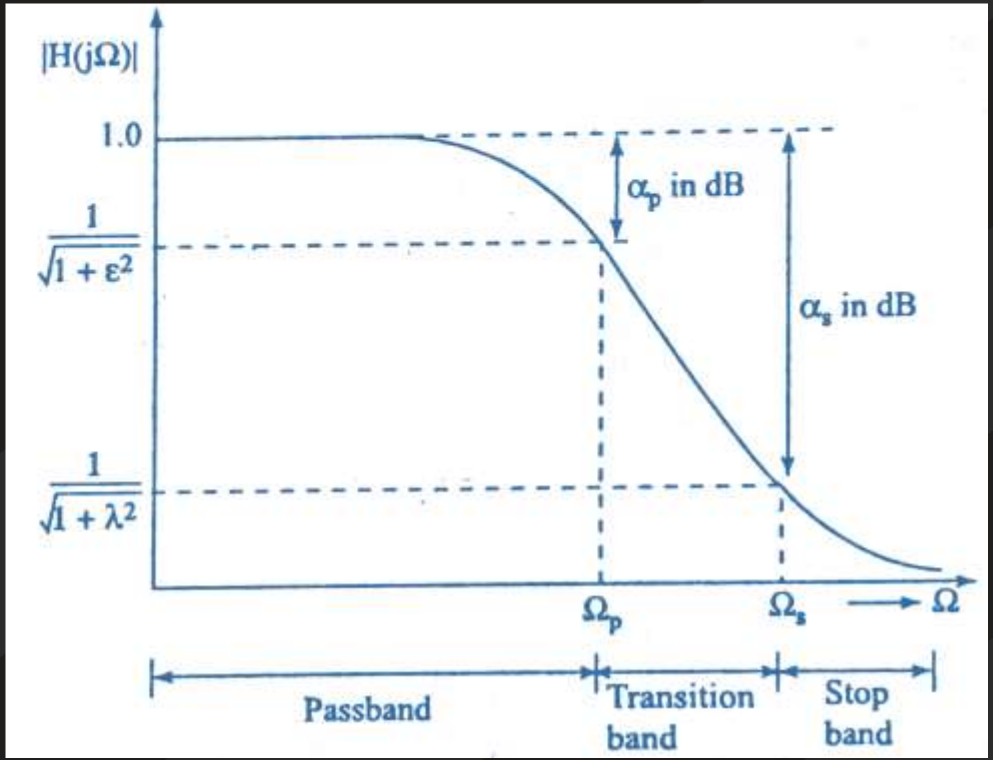
$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

$$H(s) = \frac{1}{\left(\left(\frac{s}{0.24\pi}\right)^2 + 0.765\frac{s}{0.24\pi} + 1\right)\left(\left(\frac{s}{0.24\pi}\right)^2 + 1.848\frac{s}{0.24\pi} + 1\right)}$$

$$H(s) = \frac{0.323}{(s^2 + 0.577s + 0.057\pi^2)(s^2 + 1.39s + 0.0576\pi^2)}$$

Design an analog Butterworth filter

Q) Design an analog Butterworth filter that has a -2dB passband attenuation at a frequency of 20 rad/sec and at least -10dB stopband attenuation at 30 rad/sec



Solution

$$\Omega_p = 20\, rad/sec \qquad |\alpha_p| = 2dB$$

$$\Omega_s = 30\, rad/sec \qquad |\alpha_s| = 10dB$$

Step 1: Find the order of the filter N & round off to higher integer

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log \sqrt{\frac{10^{0.1*10} - 1}{10^{0.1*2} - 1}}}{\log\left(\frac{30}{20}\right)} = 3.37 \qquad N \approx 4$$

Design an analog Butterworth filter

Step 2: Find the transfer function H(s) for Ωc=1 rad/sec for the values of N

$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2s}+1)(s^2+0.517s+1)}$

Step 3: Calculate value of cut-off frequency Ωc

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{\frac{1}{2N}}} = \frac{20}{(10^{0.1*2} - 1)^{\frac{1}{2*4}}} = 21.386$$

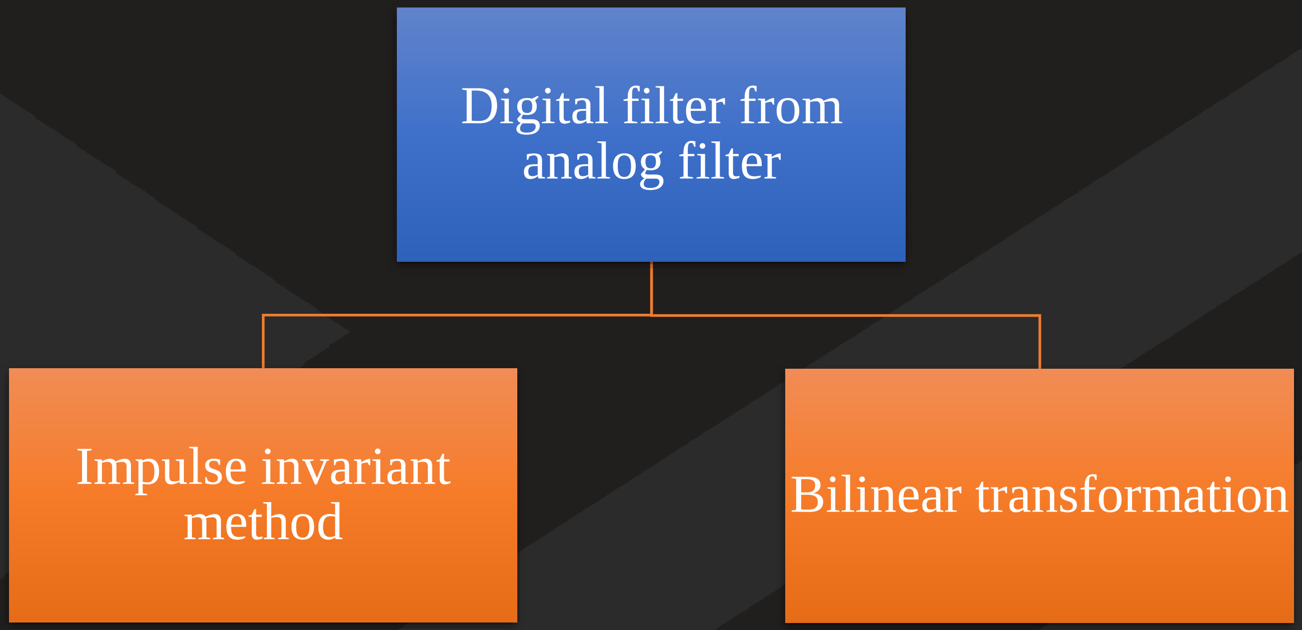
Design an analog Butterworth filter

Step 4: Find transfer function Ha(s) for the value of Ω_c calculated by substituting $s = \frac{s}{\Omega_c}$ in H(s)

$$H(S_n) = \frac{1}{\left(\left(\frac{s}{21.386}\right)^2 + 0.765\frac{s}{21.386} + 1\right)\left(\left(\frac{s}{21.386}\right)^2 + 1.848\frac{s}{21.386} + 1\right)}$$

$$H(S_n) = \frac{0.20921 \times 10^6}{(s^2 + 16.368s + 457.39)(s^2 + 39.51s + 457.394)}$$

Design digital filter from analog filter



Design of IIR filter using Impulse invariant method

- Here we require that the impulse response of the discrete system (digital filter) be the discrete version of the impulse response of the analogue system (filter)
- Hence the name impulse invariant
- In impulse invariant method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the sampled version of impulse response of analog filter

Z transform

$$H(Z) = \sum_{n=0}^{N} h(n) Z^{-n}$$

For impulse invariant method we do the mapping as

$$H(Z)|_{Z=e^{sT}} = \sum_{n=0}^{N} h(n) e^{-sTn}$$

$S = \sigma + j\Omega$ $\quad$ $Z = re^{j\omega}$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

Equating real and imaginary parts

$r = e^{\sigma T}$

Real part of analog pole =radius of Z-plane pole

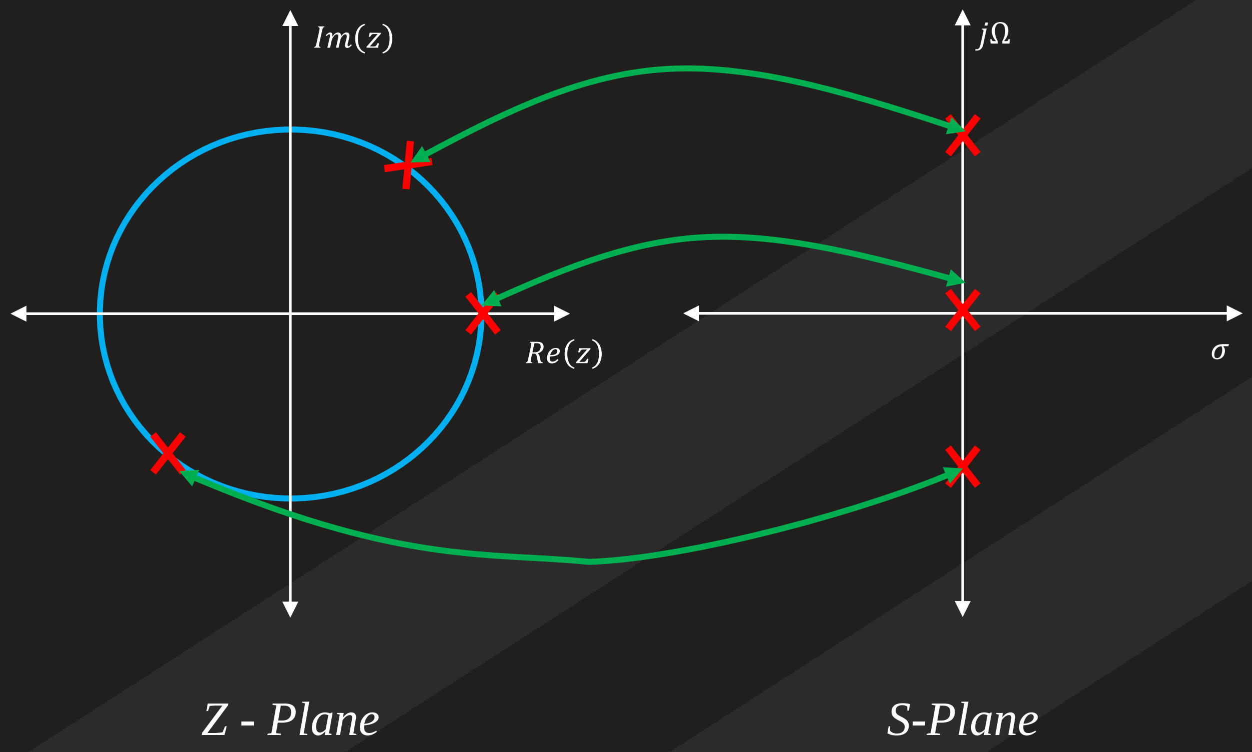
$\omega = \Omega T$

Imaginary part of analog pole = angle of digital pole

Case 1 :

$$\sigma = 0$$

$$r = e^{0T} = 1$$

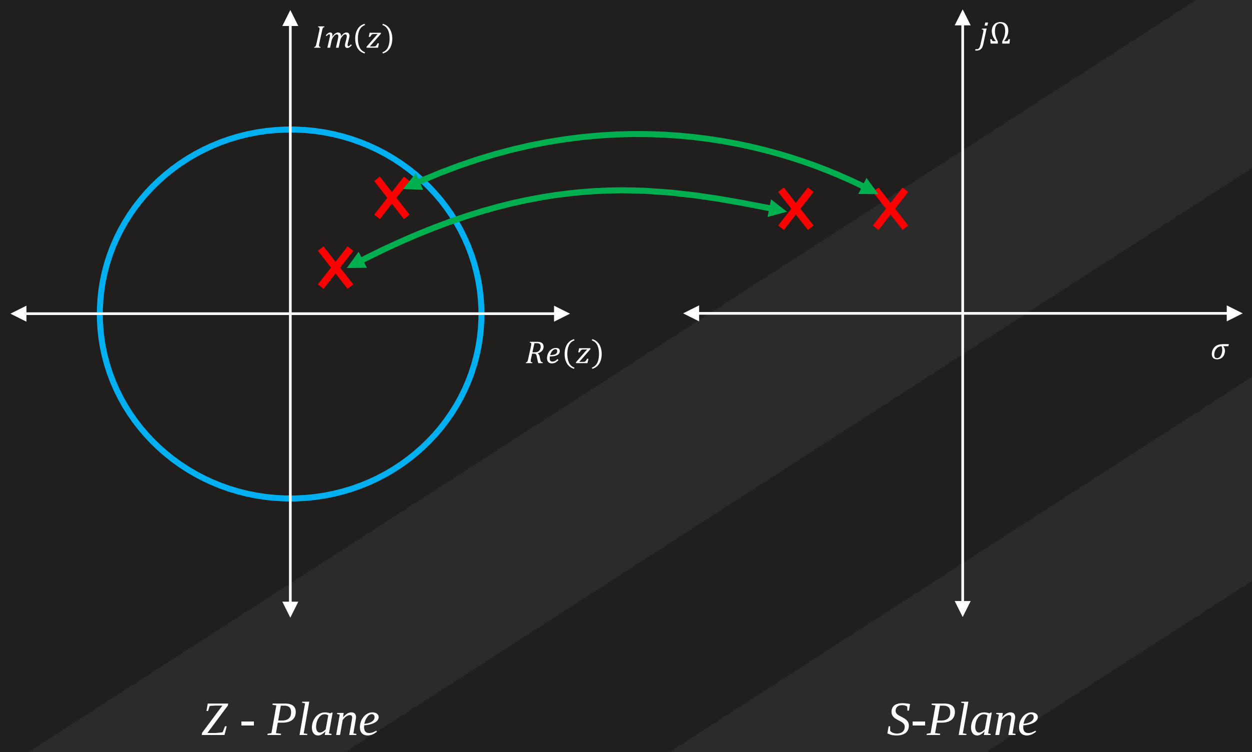


Impulse invariant mapping map poles from s-plane's jΩ axis to Z-plane's unit circle

Case 2 :

$\sigma < 0$ (poles in left half of $S-$plane)

$r = e^{\sigma T} < 1$

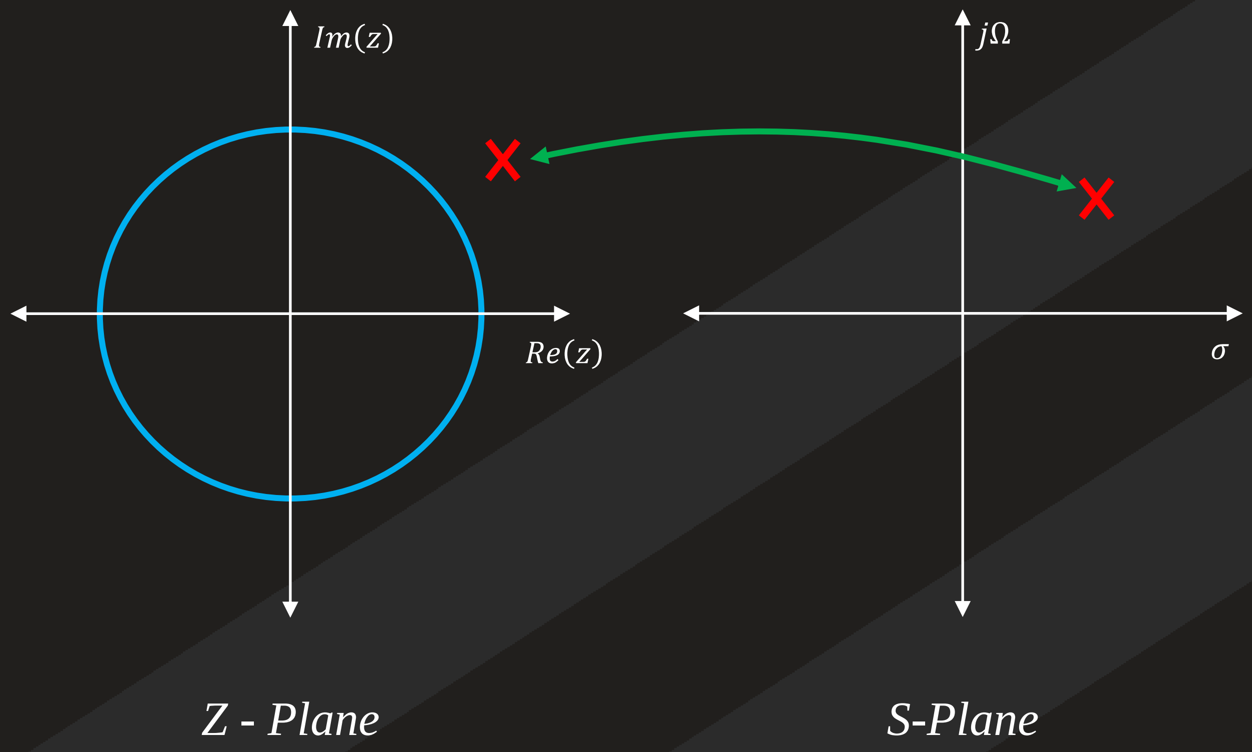


All S-plane poles with –ve real parts map to Z–plane poles inside unit circle

Case 3 :

$\sigma > 0$ (poles in right half of S – plane)

$r = e^{\sigma T} > 1$



Poles in right half of S-plane map to digital poles outside unit circle

Let $H_a(s)$ is the system function of analog filter

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$P_k \rightarrow Poles\ of\ analog\ filter$

$C_k \rightarrow Coefficients\ in\ partial fraction\ expansion$

$$L\left[\frac{1}{s-a}\right] = e^{at}$$

Taking inverse Laplace transform

$$h_a(t) = \sum_{k=1}^{N} C_k e^{P_k t}$$

Sample $h_a(t)$ at $t=nT$

$$h(n) = h(nT) = \sum_{k=1}^{N} C_k e^{P_k nT}$$

Now taking Z - Transform

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty}\sum_{k=1}^{N} C_k e^{P_k nT}\, z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} (e^{P_k T} z^{-1})^n$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

Steps to design IIR filter using Impulse invariant method

1. *For the given specification find Ha(s), the transfer function of analog filter*
2. *Select the sampling rate of the digital filter, T seconds/sample*
3. *Express the analog filter transfer function as the sum of single pole filters*

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

4. *Compute the Z transform of the digital filter by using the formula*

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{P_k T} z^{-1}} \quad \text{for T} < 1$$

Design of IIR filter by impulse invariant technique

Q) For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ Determine H(z) using impulse invariance method. Assume T=1sec

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

Solution

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Using partial fraction method

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$2 = A(s+2) + B(s+1)$$

At s = -1

$$A = 2$$

At s = -2

$$B = -2$$

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

$$H(s) = \frac{2}{(s-(-1))} - \frac{2}{(s-(-2))}$$

For T=1 sec

$$\frac{2}{(s-(-1))} = \frac{2}{1 - e^{-1}z^{-1}}$$

$$\frac{2}{(s-(-2))} = \frac{2}{1 - e^{-2}z^{-1}}$$

$$H(z) = \frac{2}{1 - e^{-1}z^{-1}} - \frac{2}{1 - e^{-2}z^{-1}}$$

$$H(z) = \frac{0.465z^{-1}}{1 - 0.503z^{-1} + 0.0497z^{-2}}$$

Design of IIR filter by impulse invariant technique

Q) An anlog filter has a transfer function $H(s) = \frac{10}{s^2+7s+10}$ Design a digital filter equivalent to this using impulse invariant method for T=0.2 sec

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

Solution

$$H(s) = \frac{10}{s^2 + 7s + 10}$$

Using partial fraction method

$$\frac{10}{s^2 + 7s + 10} = \frac{A}{(s+5)} + \frac{B}{(s+2)}$$

$$10 = A(s+2) + B(s+5)$$

At s = -5   At s = -2

$A = -3.33$ $B = 3.33$

$$H(s) = \frac{-3.33}{(s+5)} + \frac{3.33}{(s+2)}$$

$$H(s) = \frac{-3.33}{(s-(-5))} + \frac{3.33}{(s-(-2))}$$

For T=0.2 sec

$$\frac{-3.33}{(s-(-5))} = \frac{-3.33}{1 - e^{-5*0.2} z^{-1}}$$

$$\frac{3.33}{(s-(-2))} = \frac{3.33}{1 - e^{-2*02} z^{-1}}$$

$$H(z) = \left[\frac{-3.33}{1 - e^{-1} z^{-1}} - \frac{3.33}{1 - e^{-0.4} z^{-1}}\right] 0.2$$

$$H(z) = \frac{0.2012 z^{-1}}{1 - 1.0378 z^{-1} + 0.247 z^{-2}}$$

Design of IIR filter by impulse invariant technique

Q) Apply impulse invariant method and find H(z) for $H(s) = \frac{s+a}{(s+a)^2+b^2}$

$$e^{at}\cos bt \overset{L}{\Leftrightarrow} \frac{(s-a)}{(s-a)^2+b^2}$$

Solution

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

Inverse Laplace of the given function

$$h(t) = e^{-at}\cos(bt)$$

For sampling the function substitute t=nT

$$h(nT) = e^{-anT}\cos(bnT)$$

Taking Z-transform

$$H(z) = \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{jbT}z^{-1}\right)^n + \left(e^{-aT}e^{-jbT}z^{-1}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-(a-jb)T}z^{-1}\right)^n + \left(e^{-(a+jb)T}z^{-1}\right)^n$$

$$\sum_{n=0}^{\infty} a^n \Leftrightarrow \frac{1}{1-a}$$

$$H(z) = \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

Steps to design IIR filter using Bilinear transformation method

The basis operation is to convert an analogue filter H(s) into an equivalent digital filter H(z) by using bilinear approximation

1. *From the given specifications find pre-warping analog frequency using formula*

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

2. *Using the analog frequency find H(s) of the analog filter*
3. *Select the sampling rate of the digital filter, call it T seconds per sample*
4. *Substitute* $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ *into the transfer function found in step 2*

Design of IIR filter by impulse invariant technique

Q) Apply bilinear transformation to $H(s) = \frac{2}{(s+1)+(s+2)}$

Solution

$$H(s) = \frac{2}{(s+1)+(s+2)}$$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ at T=1sec

$$H(z) = \frac{2}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right)+\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$= \frac{2}{\left(2\left[\frac{[1-z^{-1}]+[1+z^{-1}]}{1+z^{-1}}\right]\right)+\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})+(2-2z^{-1}+2+2z^{-1})}$$

$$= \frac{(1+z^{-1})^2}{(3-z^{-1})2}$$

$$H(z) = \frac{(1+z^{-1})}{(1-0.33z^{-1})}$$

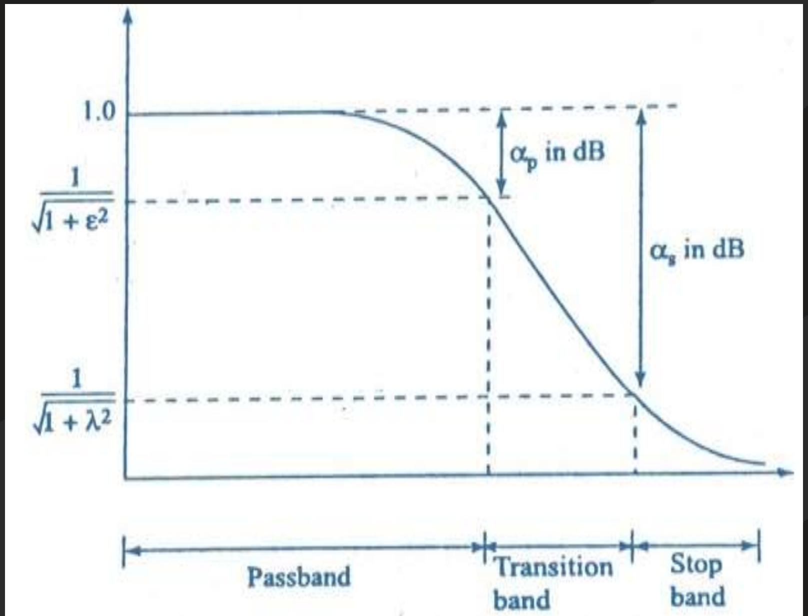
Design an analog Butterworth filter

Q) Design a digital analog Butterworth filter satisfying the constraints

$$0.707 \leq |H(j\omega)| \leq 1 \; for \; 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(j\omega)| \leq 0.2 \; for \; \frac{3\pi}{4} \leq \omega \leq \pi$$

using bilinear transformation. Take T=1sec



Solution

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707 \qquad \varepsilon = 1 \qquad \omega_p = \frac{\pi}{2}$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \qquad \lambda = 4.89 \qquad \omega_s = \frac{3\pi}{4}$$

Step 1: Step 1: From the given specifications find pre-warping analog frequency using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \qquad \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

$$\Omega_s = \frac{2}{1}\tan\frac{\frac{3\pi}{4}}{2} \qquad \Omega_p = \frac{2}{1}\tan\frac{\frac{\pi}{2}}{2}$$

$$\Omega_s = 2\tan\frac{3\pi}{8} \qquad \Omega_p = 2\tan\frac{\pi}{4}$$

$$\frac{\Omega_s}{\Omega_p} = \frac{2\tan\frac{3\pi}{8}}{2\tan\frac{\pi}{4}} = 2.414$$

Design an analog Butterworth filter

Step 2: Using the analog frequency find H(s) of the analog filter

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{4.89}{1}\right)}{\log(2.414)} \geq 1.80$$

$$N = 2$$

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} = \frac{2\tan\frac{\pi}{4}}{1^{\frac{1}{2}}} = 2\, rad/sec$$

To find H(s) substitute $s = \frac{s}{\Omega_c}$

$$H(s) = \frac{1}{\frac{s^2}{4} + \sqrt{2}\frac{s}{2} + 1}$$

$$H(s) = \frac{4}{s^2 + 2.828s + 4}$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2}s+1)(s^2+0.517s+1)}$

Design an analog Butterworth filter

Step 4: Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ *into the transfer function*

$$H(s) = \frac{1}{s^2 + 2.828s + 4}$$

$$= \frac{1}{\left[2\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 2.828\left(2\frac{1-z^{-1}}{1+z^{-1}}\right) + 4}$$

$$H(s) = \frac{4[1+z^{-1}]}{4[1-z^{-1}] + 5.656[1-z^{-2}] + 4[1+z^{-1}]}$$